COMBINATORIAL PROPERTIES, INVARIANTS, STRUCTURES AND FORMULAS ASSOCIATED WITH SOME ACTIONS OF THE ALTERNATING GROUP

RICHARD KARIUKI GACHIMU

DOCTOR OF PHILOSOPHY
(Pure Mathematics)

JOMO KENYATTA UNIVERSITY OF
AGRICULTURE AND TECHNOLOGY

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Richard Kariuki Gachimu

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DECLARATION

This thesis is my original work and has not been presented for a degree in any other University.

Signature: .... Date: ....

Richard Kariuki Gachimu

This thesis has been submitted for examination with our approval as University Supervisors.

Signature: .... Date: ....

Prof. Ireri Nthiga Kamuti
Kenyatta University, Kenya

Signature: .... Date: ....

Dr. Lewis Namu Nyaga
JKUAT, Kenya

Signature: .... Date: ....

Dr. Jane Kagwiria Rimberia
Kenyatta University, Kenya

DEDICATION

To GOD Almighty, Creator of heaven, earth and all that exists, and who goes beyond any comprehension; whatever I am, or will ever be, I owe it to GOD.

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SYMBOLS AND ABBREVIATIONS

$< g >$	The cyclic group generated by the element g
$[x_1, x_2, \cdots, x_k]$	An ordered set with k elements
$\cap$	Set intersection
$\cup$	Set union
$\emptyset$	The empty set
$\exists$	There exists
$\forall$	For all
$\frac{a}{b}$	a divided by b
∞	Infinity
$\left(\begin{array}{c} n \\ r \end{array} \right)$	n combination r
$\Longrightarrow$	Implies
$\mathbb{Q}$	The field of rational numbers
$\mathbb{Z}^+$	The set of positive integers
$\mathbb{Z}$	The set of integers
$\mid fix(g) \mid$	The number of elements fixed by a group element g
$\mid G : H \mid$	The index of H in G
$\mid X \mid$	The cardinality of a set X
${}^{a}/_{b}$	a divided by b
$\phi(n)$	The number of positive integers not exceeding n but coprime to n
$\prod$	Product
$\sum$	Summation
$\triangle$	A suborbit of a group action
$\triangle^*$	The suborbit that is paired with a suborbit $\triangle$
$\{x \in D : P(x)\}$	The set of elements x in D for which property $P(x)$ holds

$\{x \in D \mid P(x)\}$	The set of elements x in D for which property $P(x)$ holds
$\{x_1, x_2, \cdots, x_k\}$	An unordered set with k elements
${}_nC_r$	n combination r
${}_nP_r$	n permutation r
$A \cong B$	A is isomorphic to B
$a \mid b$	a divides b
$A \rtimes B$	The semidirect product of A by B
$A \setminus B$	The set of elements in A but not in B
$A \subseteq B$	A is a subset of B
$A \times B$	The Cartesian product of the set A by the set B
A_n	The alternating group of degree n
C_n	The cyclic group of degree n
D_n	The dihedral group of degree n
$H \leq G$	H is a subgroup of G
$K \lhd G$	K is a normal subgroup of G
$mon(g)$	The monomial of a group element g
O	A suborbital of a group action
$Orb_G x$	The orbit of the group G containing an element x
$P\Gamma L_2(8)$	The projective semilinear group
S_n	The symmetric group of degree n
$Stab_G(x)$	The stabilizer in the group G, of an element x
$x < y$	x is less than y
$x = y$	x is equal to y
$x > y$	x is greater than y
$x \geq y$	x is greater than or equal to y

$x \in X$	x is an element of the set X
$x \leq y$	x is less than or equal to y
$x \neq y$	x is not equal to y
$x \notin X$	x is not an element of the set X
$X^{(r)}$	The set of unordered r-element subsets of the set X
$X^{[r]}$	The set of ordered r-element subsets of the set X
$Z(G)$	The cycle index of a group G
$Z_{G,X}$	The cycle index of the group G acting on the set X
g.c.d	greatest common divisor
l.c.m	least common multiple

ABSTRACT

Various group actions have been studied in the past with respect to their associated combinatorial properties, invariants, structures and formulas. This thesis focuses on the combinatorial properties, invariants and structures of the alternating group A_n acting on $X^{[r]}$ and $X^{(r)}$, respectively the ordered and unordered r-element subsets of the set X of n letters. It also aims at deriving an expression of the cycle index of the symmetric group S_n, a semidirect product of A_n by the cyclic group C_2 of order 2, explicitly in terms of the cycle index of A_n and that of C_2. Transitivity of the actions is established using either the Cauchy-Frobenius Lemma or the Orbit-Stabilizer Theorem; primitivity is determined from the definition of blocks; ranks and subdegrees are computed using combinatorial arguments; pairing of suborbits is determined from definition; the suborbital graphs are constructed from their corresponding suborbitals; and the cycle index is derived from definition. The study shows that the action of A_n on $X^{[r]}$ is transitive and imprimitive if and only if $n \geq r+2$, while the rank is constant for all $n \geq 2(r+1)$. On the other hand, the action of A_n on $X^{(r)}$ is shown to be transitive for all $n \geq r+1$ and imprimitive if and only if $n = 2r$, while the rank is $r+1$ if and only if $n \geq 2r$. Further, the ranks and subdegrees of the two actions are calculated and pairing of the associated suborbits explored. Moreover, suborbital graphs related to the actions are seperately constructed and examined for directedness, connectedness, number of components, vertex degrees, and girths, depending on whether a corresponding suborbit is self-paired or paired with another, and also the number of elements from a fixed r-element subset that each element of the suborbit has. Finally, an expression of the cycle index of S_n, explicitly in terms of the cycle index of A_n and that of C_2, is obtained.

CHAPTER ONE
INTRODUCTION

1.1 Background Information

1.1.1 Group Actions

Definition 1.1.1. Let G be a group and X a non-empty set. Then G acts on the left of X if there exists a function $G \times X \to X$ such that $(g_1g_2)x = g_1(g_2)x$ and $1x = x$, where 1 is the identity in G, $x \in X$ and $g_1, g_2 \in G$. The action of G on the right of X can be defined in a similar way. In this case, X is called a G-set.

Definition 1.1.2. Suppose a group G acts on a set X. Define a relation $x \sim y$ on X if and only if there exists $g \in G$ such that $y = gx$. This defines an equivalence relation on X. The equivalence class containing x is $Orb_G x = \{gx|g \in G\}$, and is called the *orbit* of x. Since any set is a disjoint union of equivalence classes under an equivalence relation, it follows that if G acts on X, then X is a union of disjoint orbits.

Theorem 1.1.1. If G is a finite group and X is a finite G-set, then the number of orbits in X under G is given by $\frac{1}{|G|}\sum\limits_{g \in G}|fix(g)|$ where $fix(g) = \{x \in X|gx = x\}$ (Rose, 1978; Gardiner, 1986; Flareigh, 1994).
Theorem 1.1.1 is called the *Cauchy-Frobenius Lemma* or the *Burnside's Formula*.

Definition 1.1.3. The *stabilizer* in G of x is the subset $Stab_G x = \{g \in G|gx = x\}$ of G. It is also denoted by G_x. According to Gardiner (1986), G_x is a subgroup of G, called the *isotropy subgroup* of G. If G_x is trivial, i.e., $G_x = \{1\}$, then G is said to act *freely* on X.

Theorem 1.1.2. Let X be a G-set and let $x \in X$. Then $|Orb_G x| = |G : G_x|$, the index of G_x in G (Rose, 1978; Gardiner, 1986; Flareigh, 1994).

Theorem 1.1.2 is called the *Orbit-Stabilizer Theorem.*

Definition 1.1.4. The action of a group G on a set X is said to be *transitive* if for each $x, y \in X$, there exists $g \in G$ such that $y = gx$; in other words $Orb_G x = X$ if $x \in X$. A group which is not transitive is called *intransitive*.

Theorems 1.1.1 and 1.1.2 are used interchangeably in the determination of transitivity of a group action.

Example 1.1.1. The symmetric group S_n acts transitively on $X = \{1, 2, \cdots, n\}$. This is so because for any $x, y \in X$, there exists an element $\alpha \in S_n$ for which $y = \alpha x$. Similarly, the alternating group A_n acts transitively on $X = \{1, 2, \cdots, n\}$ for all for all $n \geq 3$, but acts intransitively if $n = 2$.

Definition 1.1.5. An action of a group G on a set X, with $|X| \geq 2$, is called 2-*transitive* (*doubly transitive*) when for any two ordered pairs of distinct elements $(x, x^{'})$ and $(y, y^{'})$ in X, there is a $g \in G$ such that $y = gx$ and $y^{'} = gx^{'}$. The distinctness of elements means $x \neq x^{'}$and $y \neq y^{'}$. The element g is said to take the pair $(x, x^{'})$ to the pair $(y, y^{'})$. One can formulate the idea of 3-*transitive* (*triply transitive*) action, and more general a k-*transitive* action for any integer $k \geq 1$: given any two ordered k-tuples $(x_1, \cdots, x_k)$ and $(y_1, \cdots, y_k)$ of distinct elements in the set, some element of the group sends x_i to y_i for all i. An action which is k-transitive is l-transitive for $l \leq k$; so any 3-transitive action is 2-transitive and transitive.

Theorem 1.1.3. The alternating group A_n has a natural $(n-2)$-transitive action on the set $X = \{1, 2, \cdots, n\}$ for all $n \geq 3$ (Smith & Tabachnikova, 2000).

Definition 1.1.6. Let G act transitively on X and let G_x be the stabilizer of $x \in X$. The orbits $\triangle_0 = \{x\}, \triangle_1, \triangle_2, \cdots, \triangle_{r-1}$ of G_x on X are known as *suborbits* of G. In this case, $\triangle_0$ is referred to as the *trivial* suborbit of G. The value r is called the *rank* and the sizes $|\triangle_i|$, $(i = 0, 1, 2, \cdots, r-1)$ the *subdegrees,* of G. Both the rank and the subdegrees of G are independent of the choice of $x \in X$ (Rose, 1978).

Definition 1.1.7. Let G act transitively on a set X and let $\triangle$ be an orbit of G_x on X. If $\triangle^* = \{gx | g \in G, x \in g\triangle\}$, then $\triangle^*$ is also an orbit of G_x called the G_x-orbit or G-suborbit *paired* with $\triangle$. Clearly, $\triangle^{**} = \triangle$ and $|\triangle| = |\triangle^*|$. If $\triangle = \triangle^*$, then $\triangle$ is said to be *self-paired*. The trivial suborbit of G is always self-paired.

Theorem 1.1.4. Let G act transitively on a set X. Then G has non-trivial self-paired suborbits if and only if G has even order (Wielandt, 1964).

Theorem 1.1.5. Let G be transitive on X and let $g \in G$. Then the number of self-paired suborbits of G is given by $\frac{1}{|G|}\sum\limits_{g \in G} |fix(g^2)|$ (Cameron, 1974).

Definition 1.1.8. Let G act transitively on a finite set X. Then a subset Y of X, where $|Y|$ is a factor of $|X|$, is called a *block* or *set of imprimitivity* for the action if for each $g \in G$, either $gY = Y$ or $gY \cap Y = \emptyset$; in other words gY and Y do not overlap partially. In particular, $\emptyset, X$ and all 1-element subsets of X are blocks, called *trivial blocks*. The action is said to be *primitive* if the only blocks are the trivial blocks, and *imprimitive* otherwise.

Theorem 1.1.6. A transitive group G on X is primitive if and only if G_x, for a fixed $x \in X$, is a maximal subgroup of G. In addition, a 2-transitive group is primitive. Moreover, any non-trivial normal subgroup of a primitive group is transitive (Cameron, 1999).

Theorem 1.1.7. Let G be transitive on X and let the subdegrees of G, ordered according to increasing magnitude, be $1 = n_0 \leq n_1 \leq n_2 \leq \cdots \leq n_{r-1}$. Then G is imprimitive if there exists an index $j > 0$ such that $n_j > n_1 n_{j-1}$ (Wielandt, 1964).

1.1.2 Graph Theory

Definition 1.1.9. A *simple graph* G consists of a non-empty set $V(G)$ of objects called *points* or *nodes* or *vertices* and a (possibly empty) set $E(G)$ of pairs of elements of V called *edges*. The set V is called the *vertex-set* and E the *edge-list*, of G.

Definition 1.1.10. Two or more edges joining the same pair of vertices are called *multiple edges* while an edge joining a vertex to itself is called a *loop*. A *multigraph* is a graph which is allowed to have multiple edges but no loops.

It is clear from definition that a simple graph has no loops or multiple edges. From this point on, by a graph it shall mean a simple graph.

Definition 1.1.11. Let v and w be two vertices of a graph G. Then an edge $e = (v, w)$ is said to *join* or *connect* v and w. In this case e is said to be *incident* with v and w and the vertices v and w are *adjacent*. A graph is called *complete* if each pair of distinct vertices is joined by an edge.

If a graph has n vertices, then the maximum number of edges it can have is $\binom{n}{2}$, the number of 2-element subsets of V.

Definition 1.1.12. Let G be a graph with vertex-set V and edge-list E. A *subgraph* of G is a graph all of whose vertices belong to V and all of whose edges belong to E.

Definition 1.1.13. The *degree* or *valency* of a vertex v of a graph G is the number of edges incident with v, and is denoted by $deg\, v$. If a graph G has n vertices, then for any vertex v, $0 \leq deg\, v \leq n - 1$. The *minimum degree* of G, denoted by $\delta(G)$, is the smallest number of edges incident with a vertex of G while the *maximum degree* of G, denoted by $\triangle(G)$, is the largest such number. The graph G is said to be *regular of degree* r if the degree of each vertex is r, i.e., if $\delta(G) = \triangle(G) = r$.

Theorem 1.1.8. In any graph, the sum of all the vertex-degrees is equal to twice the number of edges (Wilson & Watkins, 1990).

Theorem 1.1.8 is called *The Handshaking Lemma*.

Definition 1.1.14. Two graphs G and H are *isomorphic*, written $G \cong H$, if there exists a one-to-one correspondence $\alpha : V(G) \to V(H)$ such that α preserves adjacency; in other words, $(u, v) \in E(G)$ if and only if $(\alpha(u), \alpha(v)) \in E(H)$. Since $|V(G)| = |V(H)|$, any one-to-one correspondence is equivalent to a relabelling of the vertices.

Definition 1.1.15. A *walk* of length k in a graph G is a succession of k edges of G of the form $v_0v_1, v_1v_2, \cdots, v_{k-1}v_k$, denoted by $v_0v_1v_2 \cdots v_{k-1}v_k$. A walk of length k all of whose edges are different (but not necessarily all the vertices) is called a *trail* of length k. A walk of length k all of whose vertices are different is called a *path* of length k. A closed walk is a succession of edges of the form $v_0v_1, v_1v_2, \cdots, v_{r-1}v_r, v_rv_0$. A closed walk all of whose edges are different is a closed trail. A closed walk all of whose vertices are different (except the first and the last) is called a *cycle*. The length of the shortest cycle (if any) in G is called the *girth* of G. Every cycle is a closed walk, but not every closed walk is a cycle.

Definition 1.1.16. A graph G is *connected* if there is a path in G between any pair of distinct vertices, and *disconnected* otherwise. Every disconnected graph can be split up into a number of connected subgraphs called *components*.

Definition 1.1.17. A *bipartite* graph is one whose vertex-set can be split into sets A and B in such a way that each edge of the graph joins a vertex in A to a vertex in B.

Definition 1.1.18. A *tree* is a connected graph that contains no cycles. A *leaf* of a tree is a vertex of degree one. A graph all of whose components are trees is called a *forest*.

Theorem 1.1.9. If a tree has n vertices, then it has $n-1$ edges (Chetwynd & Diggle, 1995).

Definition 1.1.19. A *directed graph* or *digraph*, D, is a graph with a function which assigns each edge e an ordered pair of vertices (u, v). The vertex u is called the *tail* of e, v the *head* of e, and u, v the *ends* of e. If there is an edge with tail u and head v, then (u, v) denotes such an edge, and the edge is *directed* from u to v. The *out-degree* of a vertex v, denoted *outdeg* v, is the number of edges with tail v, and the *in-degree* of v, denoted *indeg* v, is the number of edges with head v.

Theorem 1.1.10. In any digraph, the sum of all the out-degrees and the sum of all the in-degrees are each equal to the number of directed edges (Wilson & Watkins, 1990).

Theorem 1.1.10 is called *The Handshaking Di-Lemma*.

1.1.3 Suborbitals and Suborbital Graphs

Definition 1.1.20. Suppose G acts on X. Then G acts on $X \times X$ also, by the rule that $g(x, y) = (gx, gy)$, $g \in G$, $x, y \in X$. If $O \subseteq X \times X$ is a G-orbit, then for a fixed element $x \in X$, $\triangle = \{y \in X | (x, y) \in O\}$ is a G_x-orbit. Conversely, if $\triangle \subseteq X$ is a G_x-orbit, then $O = \{(gx, gy) | g \in G, y \in \triangle\}$ is a G-orbit on $X \times X$. This means there is a one-to-one correspondence between the orbits of G_x ($x \in X$) on X and the orbits of G on $X \times X$. In this case $\triangle$ is said to correspond to O. The G-orbits on $X \times X$ are called *suborbitals*. The number of these suborbitals is equal to the rank of G.

Definition 1.1.21. Let $O_i \subseteq X \times X$, $(i = 0, 1, 2, \cdots, r-1)$ be a suborbital. Then a graph Γ_i is formed by taking X as the points of Γ_i and including a directed line from x to y, where

$x, y \in X$, if and only if $(x, y) \in O_i$. Thus each suborbital O_i determines a suborbital graph Γ_i. Now, define $O_i^* = \{(y, x) | (x, y) \in O_i\}$. Then O_i^* is a G-orbit also and determines a corresponding suborbital graph Γ_i^*.

Theorem 1.1.11. Let G be transitive on X and let Γ be the suborbital graph corresponding to the suborbit $\triangle$. Then Γ is undirected if $\triangle$ is self-paired and directed otherwise (Sims, 1967).

Theorem 1.1.12. Let G be transitive on X. Then G is primitive if and only if every non-trivial suborbital graph corresponding to the action is connected (Sims, 1967).

1.1.4 Permutation Groups

Definition 1.1.22. Let $X = \{1, 2, \cdots, n\}$. Then the *symmetric group* of degree n is the group of all permutations of X under the binary operation of composition of maps. It is denoted by S_n and has order $n!$. A subgroup of S_n is called a *permutation group*.

Definition 1.1.23. The *alternating group* of degree n is the subgroup of S_n consisting of all even permutations of X under the binary operation of composition of maps. It is denoted by A_n and has order $\frac{n!}{2}$.

Definition 1.1.24. A group G is called *cyclic* if $G = \{g^k \mid k \in \mathbb{Z}\}$, that is G can be generated by a single element $g \in G$. This is denoted by $G =< g >$. The element g in this case is called the *generator* of G. A cyclic group can have two or more generators.

Example 1.1.2. A cyclic subgroup of S_n of degree n and order n is denoted by C_n. Just but to mention,

$$C_2 =< (ab) >= \{1, (ab)\},$$

$$C_3 =< (abc) >=< (acb) >= \{1, (abc), (acb)\}$$

and

$$C_4 =< (abcd) >=< (adcb) >= \{1, (abcd), (ac)(bd), (adcb)\}.$$

The subgroup C_2 generated by a transposition will be of particular interest at some later stage of this thesis.

1.1.5 Operations on Permutation Groups

There are some important operations on permutation groups which produce other permutation groups. These include sum, product, composition, and power group. For the purpose of definitions of these operations, consider a permutation group G of order m_1 and degree n_1 defined on the set $X = \{x_1, x_2, \cdots, x_{n_1}\}$, and another permutation group H of order m_2 and degree n_2 defined on the set $Y = \{y_1, y_2, \cdots, y_{n_2}\}$.

Definition 1.1.25. The *sum* $G + H$ is a permutation group which is defined on the disjoint union $X \cup Y$ and whose elements are the ordered pairs (g, h) of permutations $g \in G$ and $h \in H$. Any element z of $X \cup Y$ is permuted by (g, h) according to the rule:

$$(g,h)z = \begin{cases} gz, & z \in X \\ hz, & z \in Y. \end{cases}$$

Clearly, $G + H$ has order $m_1 m_2$ and degree $n_1 + n_2$. The group operation between the pairs of elements of the sum is given by

$$(g,h)(g^{'},h^{'}) = (gg^{'},hh^{'}).$$

Definition 1.1.26. The *product* $G \times H$, also called the *Cartesian product* of G and H is a permutation group which is defined on the set $X \times Y$ and whose elements are all the ordered pairs (g, h) of permutations $g \in G$ and $h \in H$. The element (x, y) of $X \times Y$ is permuted by (g, h) using the rule $(g, h)(x, y) = (gx, hy)$. Clearly, $G \times H$ has order $m_1 m_2$ and degree $n_1 n_2$. The group operation between the pairs of elements of the product is given by

$$(g,h)(g^{'},h^{'}) = (gg^{'},hh^{'}).$$

Definition 1.1.27. The *composition* $G[H]$ of G around H, also called the *wreath product* of G by H, is also defined on $X \times Y$. For each $g \in G$ and any sequence $(h_1, h_2, \cdots, h_{n_1})$ of n_1 (not necessarily distinct) permutations in H, there is a unique permutation in $G[H]$ written $(g; h_1, h_2, \cdots, h_{n_1})$ such that for (x_i, y_j) in $X \times Y$, $(g; h_1, h_2, \cdots, h_{n_1})(x_i, y_j) = (gx_i, h_i y_j)$ for all $i = 1, 2, \cdots, n_1$, $j = 1, 2, \cdots, n_2$. The order of $G[H]$ is clearly $|G||H|^{n_1} = m_1 m_2^{n_1}$ while its degree is $n_1 n_2$. The group operation between the pairs of elements of the wreath product is given by $(g; h_1, h_2, \cdots, h_{n_1})(g^{'}; h_1^{'}, h_2^{'}, \cdots, h_{n_1}^{'}) = (gg^{'}; g^{'}h_1 h_1^{'}, g^{'}h_2 h_2^{'}, \cdots, g^{'}h_{n_1} h_{n_1}^{'})$.

Definition 1.1.28. The *power group* H^G is the set of permutations defined on the set Y^X of all functions from X to Y. The assumption is that the power group acts on more than one function. For each pair of permutations $g \in G$ and $h \in H$ there is a unique permutation, written h^g in H^G. The action of the permutation h^g on any function $f : X \to Y$ in H^G is given by $(h^g f)(x) = hf(gx)$ for each $x \in X$. The group operation between the pairs of elements of H^G is given by

$$(h^g)\left(h^{'g^{'}}\right) = hh^{'gg^{'}}.$$

The order of H^G is $|G||H| = m_1 m_2$, and its degree is $n_2^{n_1}$.

The sum, product, and power group are all isomorphic as abstract groups, although they are different as permutation groups, and the three operations are commutative (Harary, 1969; Harary, 1970).

1.1.6 Direct Products

The product $H \times K$ can be defined for any two groups H and K.

Definition 1.1.29. Let H and K be any two groups. The product $H \times K$ is a group with multiplication of its elements defined by $(h,k)(h^{'},k^{'}) = (hh^{'},kk^{'})$ for all $h,h^{'} \in H$ and $k,k^{'} \in K$. In this case $H \times K$ is called the *external direct product* of H and K.

The definition of direct product of two groups can be extended to the direct product of any finite number of groups.

Definition 1.1.30. Let n be any positive integer and let $G_1, G_2, \cdots, G_n$ be any n groups, which are not necessarily distinct. Then $G_1 \times G_2 \times \cdots \times G_n$ is the set of n-tuples $(g_1, g_2, \cdots, g_n)$ with $g_i \in G_i$. This set is given the structure of a group called the external direct product of the groups $G_1, G_2, \cdots, G_n$ by defining multiplication of the n-tuples component-wise; in other words if $(g_1, g_2, \cdots, g_n), (g_1^{'}, g_2^{'}, \cdots, g_n^{'}) \in G_1 \times G_2 \times \cdots \times G_n$, then

$$(g_1, g_2, \cdots, g_n)(g_1^{'}, g_2^{'}, \cdots, g_n^{'}) = (g_1g_1^{'}, g_2g_2^{'}, \cdots, g_ng_n^{'}) \in G_1 \times G_2 \times \cdots \times G_n.$$

This group is denoted by $Dr \prod\limits_{i=1}^{n} G_i$.

Definition 1.1.31. Let H and K be normal subgroups of a group G with $H \cap K = \{1\}$ and $G = HK$. Then G is a group, called the *internal direct product* of H by K. In this case $G \cong H \times K$.

Example 1.1.3. The Klein-4 group $G = \{1, (12)(34), (13)(24), (14)(23)\}$ has normal subgroups $H = \{1, (12)(34)\}$ and $K = \{1, (14)(23)\}$ such that $H \cap K = \{1\}$ and $G = HK$. Thus, G is an internal direct product of H by K, and $G \cong H \times K$.

Definition 1.1.32. Let G be a group with subgroups H and K. Then G is called a *semidirect product* of K by H, denoted $G = H \ltimes K$ or $G = K \rtimes H$, if $K \lhd G$, $H \cap K = \{1\}$ and $G = HK$.

Example 1.1.4. Consider the group $G = S_3 = \{1, (123), (132), (12), (13), (23)\}$. In this case $K = A_3 = \{1, (123), (132)\}$ and $H = C_2 = \{1, (12)\}$ are subgroups of G with $K \lhd G$. It is easy to verify that G is a semidirect product of K by H (though not an internal direct product since H is not normal in G). In fact, $S_n = A_n \rtimes C_2$ for all $n \geq 2$. Similarly, $D_n = C_n \rtimes C_2$ for all $n \geq 3$.

Definition 1.1.33. A *Frobenius group* is a group G acting on a set X, transitively, in such a way that the stabilizer H of a point is non-trivial, but only the identity fixes two or more points. That means that $H \cap (gHg^{-1}) = \{1\}$ if $g \in G \setminus H$. Define $K^* = G \setminus \cup\{gHg^{-1} : g \in G\}$, the set of all elements in G having no fixed points. Then $K = K^* \cup \{1\}$ is a normal subgroup of G. Besides, $G = K \rtimes H$.

Example 1.1.5. The group $G = S_3$ acts transitively on $X = \{1, 2, 3\}$ (see Example 1.1.1). In this case, $H = Stab_G(3) = \{1, (12)\}$, but only the identity fixes two or more points. So, S_3 is a Frobenius group, by Definition 1.1.33. Furthermore, $G \setminus H = \{(123), (132), (13), (23)\}$. In this case $gHg^{-1} = \{1, (23)\}$ if $g = (123)$ or $g = (13)$, $gHg^{-1} = \{1, (13)\}$ if $g = (132)$ or $g = (23)$, and $gHg^{-1} = H$ if $g \in H$. Accordingly, $H \cap (gHg^{-1}) = \{1\}$ whenever $g \in G \setminus H$, and $M = \cup\{gHg^{-1} : g \in G\} = \{1, (12), (13), (23)\}$. So, if $K^* = G \setminus M = \{(123), (132)\}$, then $K = K^* \cup \{1\} = \{1, (123), (132)\} = A_3$, and from Example 1.1.4, $G = K \rtimes H$.

In general, if $n \geq 4$, the stabilizer of a point of $X = \{1, 2, \cdots, n\}$ in S_n has non-trivial elements which fix more than one element. Therefore, S_n is not Frobenius for $n \geq 4$.

1.1.7 Cycle Indices

Definition 1.1.34. If a finite group G acts on a set X with n elements, each $g \in G$ corresponds to a permutation σ of X, which can be written uniquely as a product of disjoint cycles. If σ has α_1 cycles of length 1, α_2 cycles of length $2, \cdots, \alpha_n$ cycles of length n, then σ and hence g has *cycle type* $(\alpha_1, \alpha_2, \cdots, \alpha_n)$.

Theorem 1.1.13. Let g be a permutation with cycle type $(\alpha_1, \alpha_2, \cdots, \alpha_n)$. Then the number of 1-cycles in g^l is $\sum\limits_{i|l} i\alpha_i$ (Bruijn & Klarner, 1969).

Theorem 1.1.14. Two permutations in S_n are conjugate if and only if they have the same cycle type; and if $g \in S_n$ has cycle type $(\alpha_1, \alpha_2, \cdots, \alpha_n)$, then the number of permutations in S_n conjugate to g is $\frac{n!}{\prod\limits_{i=1}^{n} \alpha_i! i^{\alpha_i}}$ (Krishnamurthy, 1985).

Definition 1.1.35. If a group G acts on a set X, $|X| = n$, and $g \in G$ has cycle type $(\alpha_1, \alpha_2, \cdots, \alpha_n)$, the *monomial* of g is defined as $mon(g) = t_1^{\alpha_1} t_2^{\alpha_2} \cdots t_n^{\alpha_n}$, where $t_1, t_2, \cdots, t_n$ are distinct commuting indeterminates.

Definition 1.1.36. The *cycle index* of the action of G on X is the polynomial (say over the rational field $\mathbb{Q}$) in $t_1, \cdots, t_n$ given by

$$Z(G) = Z_{G,X}(t_1, \cdots, t_n) = \frac{1}{|G|} \sum_{g \in G} \{mon(g)\}.$$

If G has conjugacy classes $K_1, \cdots, K_m$ with $g_i \in K_i$, then

$$Z(G) = \frac{1}{|G|} \sum_{i=1}^{m} |K_i| \{mon(g_i)\}. \tag{1.1.1}$$

Example 1.1.6. Let $X = \{1, 2, 3, 4\}$ so that $G = S_4$. Then,

$$\begin{aligned} G &= \{1, (123), (132), (124), (142), (134), (143), (234), (243), (12)(34), \\ &\quad (13)(24), (14)(23), (12), (13), (14), (23), (24), (34), (1234), \\ &\quad (1243), (1324), (1342), (1423), (1432)\}. \end{aligned}$$

Now, Table 1.1 below displays the various permutation types of elements of G, and their corresponding cycle types and monomials. The number of elements with the same cycle type can be counted directly from G and are given in the fourth column of the table (it is only reasonable to calculate this number using the expression in Theorem 1.1.14 for cases where $|G|$ is relatively large).

Table 1.1: Monomials of Elements of S_4

Permutation Type	Cycle Type	Corresponding Monomial	Corresponding Number of Elements in S_4
$(a)(b)(c)(d)$	$(4, 0, 0, 0)$	t_1^4	1
$(a)(bcd)$	$(1, 0, 1, 0)$	t_1t_3	8
$(ab)(cd)$	$(0, 2, 0, 0)$	t_2^2	3
$(a)(b)(cd)$	$(2, 1, 0, 0)$	$t_1^2t_2$	6
$(abcd)$	$(0, 0, 0, 1)$	t_4	6
Total			$24 = |S_4|$

Lastly, from Table 1.1 and Equation 1.1.1, $Z_{G,X} = \frac{1}{24}\{t_1^4 + 8t_1t_3 + 3t_2^2 + 6t_1^2t_2 + 6t_4\}$.

Definition 1.1.37. Let A and B be polynomials in the indeterminates $t_1, t_2, \cdots$. Then, by the *Plethysm* of A and B, it means the replacing of every t_i in A ($i = 1, 2, \cdots$) by the polynomial obtained from B by multiplying by i the subscript of each of its indeterminates. The resulting polynomial is denoted by $A[B]$.

Example 1.1.7. Consider $Z(S_2) = \frac{1}{2}\{t_1^2 + t_2\}$ and $Z(S_3) = \frac{1}{6}\{t_1^3 + 2t_3 + 3t_1t_2\}$, the cycle indices of S_2 and S_3 respectively. Then $Z(S_2[S_3])$ is obtained by replacing t_1 in $Z(S_2)$ by $\frac{1}{6}\{t_1^3 + 2t_3 + 3t_1t_2\}$ and t_2 in $Z(S_2)$ by $\frac{1}{6}\{t_2^3 + 2t_6 + 3t_2t_4\}$. Thus

$$Z(S_2[S_3]) = \frac{1}{72}\{t_1^6 + 4t_3^2 + 9t_1^2t_2^2 + 4t_1^3t_3 + 12t_1t_2t_3 + 6t_1^4t_2 + 6t_2^3 + 12t_6 + 18t_2t_4\}.$$

1.2 Statement of the Problem

The current study seeks to answer the open question about transitivity, primitivity, ranks, subdegrees, pairing of suborbits, and properties of the suborbital graphs, associated with the actions of A_n on $X^{[r]}$ and $X^{(r)}$, respectively the ordered and unordered r-element subsets of the set $X = \{1, 2, \cdots, n\}$. In addition, it focuses on deriving an expression of the cycle index of S_n, explicitly in terms of the cycle index of A_n and that of C_2.

1.3 Justification

The theory of combinatorics has applications in counting the number of distinct objects in the presence of symmetry. Applications of this theory is present in the enumeration of graphs and solving a lot of counting problems connected with chemical isomers. The two main theorems used in this field are the Burnside's Formula, which utilizes the concept of orbits to count mathematical objects with regard to symmetry, and the Polya's Enumeration Theorem, which uses the cycle index of a group.

Graph theory has applications in modelling types of relations and process dynamics in physical, biological, social and information systems. Many practical problems can be represented by graphs. For instance, graph theory is important in chemistry and physics when studying the nature of bonding in crystals. In computer science, graphs are used to represent networks of communication, data organization, computation devices, the flow of computation, etc. One practical example is the link structure of a website which could be represented as a directed graph, where the vertices are the web pages available at the website and a directed edge from vertex A to vertex B exists if and only if page A contains a link to page B. In addition, homogeneous multiprocessor systems are usually modelled by undirected graphs.

1.4 Objectives

1.4.1 General Objective

To determine combinatorial properties, invariants, structures, and formulas, associated with some actions of the alternating group A_n.

1.4.2 Specific Objectives

1. To determine transitivity and primitivity of the actions of A_n on $X^{[r]}$ and $X^{(r)}$.
2. To compute the ranks and subdegrees of the actions of A_n on $X^{[r]}$ and $X^{(r)}$.
3. To examine pairing of the suborbits corresponding to the actions of A_n on $X^{[r]}$ and $X^{(r)}$.
4. To construct the suborbital graphs associated with the actions of A_n on $X^{[r]}$ and $X^{(r)}$ and to analyse the theoretic properties of these graphs.
5. To derive an expression of the cycle index of the semidirect product $S_n = A_n \rtimes C_2$ explicitly in terms of the cycle indices of A_n and C_2.

1.5 Null Hypothesis

The actions of A_n on $X^{[r]}$ and $X^{(r)}$ are not transitive and the cycle index formula of S_n is not explicitly expressible in terms of the cycle indices of A_n and C_2.

CHAPTER TWO
LITERATURE REVIEW

2.1 Introduction

The concepts of combinatorial properties, invariants, structures and formulas associated with a group action form the basis for the study of the group action. This chapter provides a review of some past studies, with respect to these concepts, that are closely related to the current study. In this regard, Section 2.2 examines studies concerned with the combinatorial properties, invariants, and structures, while Section 2.3 examines studies dealing with combinatorial formulas, of different group actions.

2.2 Combinatorial Properties, Invariants and Structures

A study by Nagai (1961) used the action of A_7 on $X^{(2)}$, the set of unordered pairs from $X = \{1, 2, \cdots, 7\}$, to illustrate the existence of a primitive not doubly transitive group of degree 21 which contains a non-abelian regular subgroup of order 21.

According to Higman (1970) the symmetric group S_n acts on $X^{(2)}$ with rank 3 and subdegrees $1,\ 2(n-2),\ \binom{n-2}{2}$. The action is imprimitive if $n = 4$ and primitive if $n \geq 5$. Higman also showed that any rank 3 group of odd order is primitive. Moreover, if $G \leq S_n$, $n \geq 4$, acts with rank 3 on $X^{(2)}$, then G is 4-transitive except for $n = 9$ and the case where $G \cong P\Gamma L_2(8)$.

According to Cameron (1981) if G is a primitive permutation group of degree $n = 2p$ where p is prime, then G has rank at most 3; and if $n-1$ is not a square, then G is 2-transitive. Moreover, if G is a permutation group of degree $\frac{1}{2}n(n-1)$ in which G_x has orbit lengths 1, $2(n-2)$, and $\frac{1}{2}(n-2)(n-3)$, and if n is not one of a known finite list of exceptional values (which includes $n = 9$), then $G \leq S_n$ (acting on unordered pairs). Cameron further proved that all finite 2-transitive groups are known and that none, except S_n and A_n, is 6-transitive.

Cameron (1999) established that if G is a primitive permutation group on X of rank r and G_x has an orbit of size $k > 1$, then

$$|X| \leq \begin{cases} 2r-1 & if\ k=2 \\ 1+\frac{k((k-1)^{r-1}-1)}{k-2} & if\ k>2. \end{cases}$$

Further, Cameron showed that the equality holds only in the cases where $r = 2$ so that G is 2-transitive, or $k = 2$ so that G is dihedral of prime degree, or $r = 3$ and $k = 3$ so that G is S_5 or A_5 of degree 5, or $r = 3$ and $k = 7$ so that G is of order 50.

Li *et al*. (2004) studied finite primitive permutation groups with a small suborbit. They first produced a precise list of primitive permutation groups with a suborbit of length 4. In particular, they showed that there exist no examples of such groups with a point stabilizer of order

$2^4 3^6$, answering a question that has been open since 1970s. They then analyzed the suborbital graphs of primitive permutation groups with a suborbit of length 3 or length 4.

Hamma & Aliyu (2010) proved that a dihedral group of degree 2^r $(r \geq 2)$ acts both transitively and imprimitively.

Hamma & Audu (2010) showed that if G_1 is a dihedral group of prime degree $p \geq 3$ and G_2 is a Sylow $p-$subgroup of G_1, then both G_1 and G_2 are transitive and primitive. They however showed that if the degree of G_1 is p^2 for a prime p, then both G_1 and G_2 act imprimitively.

Rimberia (2011); Rimberia *et al.* (2012, 2013) investigated the action of S_n on $X^{[r]}$, the set of ordered r-element subsets of $X = \{1, 2, \cdots, n\}$. It was proved that the action is both transitive and imprimitive. The rank of the action was found to be equal to $(r!)^2 \sum\limits_{i=0}^{r} \frac{1}{(i!)^2 (r-i)!}$ for all $n \geq 2r$, and the corresponding subdegrees were also calculated. Further, it was shown that suborbital graphs of the action, corresponding to self-paired suborbits, have girth zero if $n \geq 2r$, while those corresponding to paired suborbits have girth three for $n \geq 3r$.

The action of S_n on $X^{(r)}$, the set of unordered r-element subsets of $X = \{1, 2, \cdots, n\}$, was studied by (Nyaga, 2012; Nyaga *et al.*, 2012; Nyaga & Kamuti, 2013). It was proved that the action is transitive. In addition, if $n \geq 2r$, the rank of the action was found to be $r + 1$ in which case the length of a suborbit $\triangle_{r-i}$ $(i = 0, 1, 2, \cdots, r)$ whose each element contains exactly i elements from the set $\{1, 2, \cdots, r\}$ is $\binom{r}{i}\binom{n-r}{r-i}$. Moreover, it was observed that the suborbits of the action are self-paired and that the suborbital graphs of the action have girth three if $n \geq 3r$.

Kamuti *et al.* (2012) investigated some properties of Γ_∞ (the stabilizer of ∞ in the modular group Γ) acting on the set $\mathbb{Z}$ of integers. They showed that the action is simply transitive and imprimitive. Additionally, they gave conditions for the orbits of the action to be paired and for the graphs associated with the action to be connected, as well as a formula for the number of connected components in a disconected graph.

Mwai (2015) investigated the transitivity, primitivity, ranks, subdegrees, and properties of suborbital graphs associated with the actions of the cyclic group C_n and the dihedral group D_n on the vertices of a regular n-gon thus extending the work of (Hamma & Aliyu, 2010; Hamma & Audu, 2010) to the general degree n for both C_n and D_n.

2.3 Combinatorial Formulas

The cycle index polynomial of S_n is $Z(S_n) = \frac{1}{n!}\sum\limits_{i=1}^{n} \frac{n!}{\prod\limits_{i=1}^{n} i^{\alpha_i}\alpha_i!}\prod\limits_{i=1}^{n} t_i^{\alpha_i}$. This can be found in (Harary, 1955; Harary, 1958; Palmer, 1973; Krishnamurthy, 1985; Bjorge, 2009; Badar & Iqbal, 2010).

Harary (1955) derived the cycle index polynomials for the pair group $S_n^{(2)}$ and the reduced ordered pair group $S_n^{[2]}$, the groups induced when the symmetric group S_n acts respectively on unordered and ordered pairs from the set $X = \{1, 2, \cdots, n\}$. These polynomials are used extensively in enumerating various types of graphs and digraphs as seen in (Harary, 1955; Harary, 1958; Harary, 1970; Harary & Palmer, 1966).

The formula for $Z(S_n^{S_2})$ was calculated by (Harary, 1958) and according to (Palmer & Robinson, 1973) $Z(S_2^{S_n})$ was calculated by Slepian. Harison & High (1968) constructed an algorithm for finding $Z(G^{S_n})$ for a general permutation group G and used their results to enumerate Post functions.

The problem of the cycle index of a general power group was solved by (Harary & Palmer, 1966). In addition, Palmer & Robinson (1973) verified an explicit general formula for $Z(G_2^{G_1})$ in terms of $Z(G_1)$ and $Z(G_2)$ for any permutation groups G_1 and G_2. The result was obtained by substituting certain operators for the variables of $Z(G_1)$ and then letting them act on $Z(G_2)$. Several applications were then sketched, including the enumeration of Boolean functions, bicolored graphs, and Post functions.

Harary (1970); Krishnamurthy (1985) showed that the cycle index of the sum $G_1 + G_2$ of two groups G_1 and G_2 is the product of $Z(G_1)$ by $Z(G_2)$; in other words
$Z(G_1 + G_2) = Z(G_1)Z(G_2)$. They also showed that the cycle index of the Cartesian product $G_1 \times G_2$ is

$$Z(G_1 \times G_2) = \frac{1}{|G_1||G_2|} \sum_{\substack{g_1 \in G_1 \\ g_2 \in G_2}} \prod_{r,s=1}^{n_1,n_2} t_{[r,s]}^{(r,s) j_r^{(g_1)} j_s^{(g_2)}},$$

where $j_r^{(g_1)}$ and $j_s^{(g_2)}$ are respectively the number of cycles of length r and s in g_1 and g_2 respectively, while $[r, s]$ and (r, s) are, respectively, the l.c.m and g.c.d of r and s, where $1 \leq r \leq n_1, 1 \leq s \leq n_2$.

Palmer & Robinson (1973); Krishnamurthy (1985) studied the cycle index of a wreath product. They showed that if $Z(G_1) = Z_{G_1,X}(t_1, \cdots, t_{n_1})$ and $Z(G_2) = Z_{G_2,Y}(s_1, \cdots, s_{n_2})$, then $Z(G_1[G_2]) = Z(G_1)[Z(G_2)]$, the Plethysm of the cycle indices of G_1 and G_2; in other words, $Z(G_1[G_2])$ is obtained by replacing each variable t_i of $Z(G_1)$ by the polynomial $Z_i(G_2)$ generated from $Z(G_2)$ by multiplying each subscript by i. This expression was originally done by Polya and it plays a key role in the enumeration of k-colored graphs and non-separable graphs.

The extension of the cycle index of $S_n^{(2)}$ to that of $S_n^{(3)}$ was done by (Palmer, 1973). Palmer used the results to calculate the number of 2-plexes for $n \leq 9$ and went further to derive the cycle index of $S_n^{(r)}$.

Grove (1983); Krishnamurthy (1985); Bjorge (2009); Badar & Iqbal (2010) showed that the cycle index of the cyclic group C_n is given as $Z(C_n) = \frac{1}{n}\sum_{i|n} \phi(i) t_i^{n/i}$, where ϕ is the Euler's phi function given by $\phi(n) = |\{d : 1 \leq d \leq n, gcd(d, n) = 1\}|$. They also expressed the cycle index polynomial of the dihedral group D_n in terms of $Z(C_n)$ as

$$Z(D_n) = \frac{1}{2}Z(C_n) + \begin{cases} \frac{1}{2}t_1 t_2^{\frac{(n-1)}{2}} & if\, n\, is\, odd \\ \frac{1}{4}\left(t_2^{n/2} + t_1^2 t_2^{\frac{(n-2)}{2}}\right) & if\, n\, is\, even. \end{cases}$$

Krishnamurthy (1985); Bjorge (2009) expressed the cycle index of A_n in terms of $Z(S_n)$ as $Z(A_n) = Z(S_n) + Z(S_n)(t_1, -t_2, t_3, -t_4, \cdots)$.

Kamuti & Obon'go (2002) extended the cycle index of $S_n^{[2]}$ to that of $S_n^{[3]}$. They also used similar techniques to derive the cycle index of $S_n^{(3)}$ which according to (Palmer, 1973) had earlier been found by Oberschelp.

Kamuti & Njuguna (2004) derived in detail the cycle index formula of $S_n^{[4]}$ and gave an outline of how that of $S_n^{[r]}$ can be obtained in general, thereby extending some results of (Harary, 1955; Kamuti & Obon'go, 2002).

Kamuti (2004) expressed the cycle index formula of a semidirect product $G = M \rtimes H$ explicitly in terms of the cycle indices of M and H. The study solved the problem by considering the cycle indices of some important semidirect products, namely the Frobenius groups. The method of solution exploited the well-known structure of Frobenius groups.

Kamuti (2012) extended a previous study by (Kamuti, 2004) by considering the internal direct product $G = M \times H$ with the aim of expressing the cycle index of G in terms of the cycle indices of M and H if G acts on the cosets of H in G.

The studies outlined in Section 2.2 above show that very little has been done on transitivity, primitivity, ranks, subdegrees, pairing of suborbits, as well as properties of the suborbital graphs associated with the actions of A_n on $X^{[r]}$ and $X^{(r)}$. Additionally, from Section 2.3, it is clear that the expression of the cycle index of the semidirect product $S_n = A_n \rtimes C_2$, explicitly in terms of the cycle indices of A_n and C_2, remains undetermined.

CHAPTER THREE
PROPERTIES AND INVARIANTS OF THE ACTION OF $\mathbf{A_n}$ ON ORDERED SUBSETS

3.1 Introduction

The combinatorial properties and invariants of a group action are fundamental concepts as far as the study of the group action is concerned. This chapter is reserved for the exploration of these concept with respect to the action of A_n on $X^{[r]}$, the set of the ordered r-element subsets of $X = \{1, 2, \cdots, n\}$. Transitivity and primitivity of the action are determined, respectively, in Sections 3.2 and 3.3. On the other hand, calculation of the rank and subdegrees of the action is done in Section 3.4, while pairing of the suborbits corresponding to the action is explored in Section 3.5.
Throughout this, and the next three chapters, for convenience, by G it shall mean the alternating group A_n.
Now, the action of G on $X = \{1, 2, \cdots, n\}$ induces an action of G on $X^{[r]}$. The induced action is defined by $g[x_1, x_2, \cdots, x_r] = [g(x_1), g(x_2), \cdots, g(x_r)]\ \forall g \in G$, $[x_1, x_2, \cdots, x_r] \in X^{[r]}$. In this case, $|X^{[r]}| = {}_nP_r = \frac{n!}{(n-r)!}$.

3.2 Transitivity of $\mathbf{A_n}$ on $\mathbf{X^{[2]}}$, $\mathbf{X^{[3]}}$ and $\mathbf{X^{[r]}}$

3.2.1 Transitivity of $\mathbf{A_n}$ on $\mathbf{X^{[2]}}$

3.2.1.1 Transitivity of $\mathbf{A_3}$ on $\mathbf{X^{[2]}}$

In this case $G = \{1, (123), (132)\}$ and $X^{[2]} = \{[1, 2], [2, 1], [1, 3], [3, 1], [2, 3], [3, 2]\}$. The identity in G fixes every element of $X^{[2]}$; the other two permutations move each element of $X^{[2]}$. So, by Theorem 1.1.1, the number of orbits corresponding to the action is

$$\frac{1}{|G|}\sum_{g\in G}|fix(g)| = \frac{1}{3}[6 + 0 + 0] = 2.$$

These are $Orb_G[1, 2] = \{[1, 2], [2, 3], [3, 1]\}$ and $Orb_G[2, 1] = \{[2, 1], [3, 2], [1, 3]\}$. Hence, by Definition 1.1.4, the action of A_3 on $X^{[2]}$ is intransitive.

3.2.1.2 Transitivity of $\mathbf{A_4}$ on $\mathbf{X^{[2]}}$

In this case,

$$\begin{aligned} G = \{&1, (123), (132), (124), (142), (134), (143), (234), \\ &(243), (12)(34), (13)(24), (14)(23)\} \end{aligned}$$

and

$$\begin{aligned} X^{[2]} &= \{[1,2],[2,1],[1,3],[3,1],[1,4],[4,1],[2,3], \\ &\qquad [3,2],[2,4],[4,2],[3,4],[4,3]\}. \end{aligned}$$

Just like in Subsubsection 3.2.1.1, only the identity element in G fixes an element of $X^{[2]}$. So, by Theorem 1.1.1, the number of orbits of the action is

$$\frac{1}{|G|}\sum_{g\in G}|fix(g)| = \frac{1}{12}[12+0+\cdots+0] = 1.$$

Therefore, by Definition 1.1.4, the action is transitive.

3.2.1.3 Transitivity of $\mathbf{A_n}$ on $\mathbf{X^{[2]}}$ for $\mathbf{n \geq 4}$

Proposition 3.2.1. The action of G on $X^{[2]}$ is transitive if and only if $n \geq 4$.

Proof. Suppose $n \geq 4$. Let $H = Stab_G[x,y]$ where $[x,y] \in X^{[2]}$. An element $g \in H$ fixes $[x,y]$ so that $g(x) = x$ and $g(y) = y$. This happens only if each of x and y belongs to a 1-cycle of g so that H is isomorphic to A_{n-2}. Hence $|H| = \frac{(n-2)!}{2}$, which makes sense since $n \geq 4$ by hypothesis. Now, by Theorem 1.1.2,

$$\begin{aligned} |Orb_G[x,y]| &= |G:H| \\ &= \frac{|G|}{|H|} \\ &= \frac{n!/2}{(n-2)!/2} \\ &= \frac{n!}{(n-2)!} \\ &= |X^{[2]}|. \end{aligned}$$

So, the action has a unique orbit and is therefore transitive. On the other hand, suppose $n < 4$. Then $(n-2)! < 2$ so that $|G| = \frac{n!}{2} < \frac{n!}{(n-2)!} = |X^{[2]}|$. Consequently, by Theorem 1.1.2, $|Orb_G[x,y]| = \frac{|G|}{|H|} \leq |G| < |X^{[2]}|$, and the action is thus intransitive. ☐

3.2.2 Transitivity of $\mathbf{A_n}$ on $\mathbf{X^{[3]}}$

3.2.2.1 Transitivity of $\mathbf{A_4}$ on $\mathbf{X^{[3]}}$

In this case, $|X^{[3]}| = {}_4P_3 = 24$. By Theorem 1.1.1, the number of orbits of the action is

$$\frac{1}{|G|}\sum_{g\in G}|fix(g)| = \frac{1}{12}[24+0+\cdots+0] = 2.$$

The two orbits of the action are

$$\begin{aligned} Orb_G[1,2,3] &= \{[1,2,3],[1,3,4],[1,4,2],[2,1,4],[2,3,1],[2,4,3],[3,1,2], \\ &\quad [3,2,4],[3,4,1],[4,1,3],[4,2,1],[4,3,2]\} \end{aligned}$$

and

$$\begin{aligned} Orb_G[1,3,2] &= \{[1,2,4],[1,3,2],[1,4,3],[2,1,3],[2,3,4],[2,4,1],[3,1,4], \\ &\quad [3,2,1],[3,4,2],[4,1,2],[4,2,3],[4,3,1]\}. \end{aligned}$$

Hence, the action is intransitive.

3.2.2.2 Transitivity of $\mathbf{A_n}$ on $\mathbf{X^{[3]}}$ for $\mathbf{n \geq 5}$

Proposition 3.2.2. The action of G on $X^{[3]}$ is transitive if and only if $n \geq 5$.

Proof. Suppose $n \geq 5$. Let $H = Stab_G[x,y,z]$ for $[x,y,z] \in X^{[3]}$. If $g \in H$, then it fixes $[x,y,z]$, which follows that $g(x) = x$, $g(y) = y$ and $g(z) = z$. This is the case only if each of the elements x, y and z belongs to a 1-cycle of g. Hence, $H \cong A_{n-3}$ and $|H| = \frac{(n-3)!}{2}$, which is practical since, by assumption, $n \geq 5$. By Theorem 1.1.2,

$$\begin{aligned} |Orb_G[x,y,z]| &= |G : H| \\ &= \frac{|G|}{|H|} \\ &= \frac{n!/2}{(n-3)!/2} \\ &= \frac{n!}{(n-3)!} \\ &= |X^{[3]}|. \end{aligned}$$

So, the action is transitive. Now, if $n < 5$, then $(n-3)! < 2$ so that $|G| = \frac{n!}{2} < \frac{n!}{(n-3)!} = |X^{[3]}|$. Accordingly, $|Orb_G[x,y,z]| = \frac{|G|}{|H|} \leq |G| < |X^{[3]}|$, by Theorem 1.1.2, and the action is therefore intransitive. $\square$

3.2.3 Transitivity of $\mathbf{A_n}$ on $\mathbf{X^{[r]}}$

Lemma 3.2.1. Let G act on $X^{[r]}$ with $n \geq r+2$. If $[x_1, x_2, \cdots, x_r] \in X^{[r]}$, then

$$|Stab_G[x_1, x_2, \cdots, x_r]| = \frac{(n-r)!}{2}.$$

Besides, $Stab_G[x_1, x_2, \cdots, x_r]$ is trivial if $n = r + 2$, so that G acts freely on $X^{[r]}$, and is non-trivial otherwise.

Proof. Let G act on $X^{[r]}$ and let $[x_1, x_2, \cdots, x_r] \in X^{[r]}$. Then $g \in G$ fixes $[x_1, x_2, \cdots, x_r]$ only if each element of $\{x_1, x_2, \cdots, x_r\}$ is in a 1-cycle of g. Hence, $Stab_G[x_1, x_2, \cdots, x_r]$ is the group of all even permutations of the set $\{x_{r+1}, x_{r+2}, \cdots, x_n\}$. But, this group is isomorphic to A_{n-r}. Therefore,

$$|Stab_G[x_1, x_2, \cdots, x_r]| = \frac{(n-r)!}{2}.$$

Now, if $n = r + 2$, then on rewriting $n - r = 2$, so that

$$|Stab_G[x_1, x_2, \cdots, x_r]| = \frac{2!}{2} = 1.$$

Hence the action of G on $X^{[r]}$ is free. On the other hand, if $n > r + 2$, then $n - r > 2$, on rewriting, so that

$$|Stab_G[x_1, x_2, \cdots, x_r]| = \frac{(n-r)!}{2} > \frac{2!}{2} = 1$$

and $Stab_G[x_1, x_2, \cdots, x_r]$ is thus non-trivial. □

Theorem 3.2.1. The action of G on $X^{[r]}$ is transitive if and only if $n \geq r + 2$.

Proof. Suppose $n \geq r + 2$. From Theorem 1.1.2, and Lemma 3.2.1,

$$\begin{aligned} |Orb_G[x_1, x_2, \cdots, x_r]| &= |G : Stab_G[x_1, x_2, \cdots, x_r]| \\ &= \frac{|G|}{|Stab_G[x_1, x_2, \cdots, x_r]|} \\ &= \frac{n!/2}{(n-r)!/2} \\ &= \frac{n!}{(n-r)!} \\ &= |X^{[r]}|, \end{aligned}$$

and then action is transitive. Now, suppose $n < r + 2$. It follows that $(n - r)! < 2$ and $|G| = \frac{n!}{2} < \frac{n!}{(n-r)!} = |X^{[r]}|$. By Theorem 1.1.2, $|Orb_G[x_1, x_2, \cdots, x_r]| = \frac{|G|}{|H|} \leq |G| < |X^{[r]}|$, and the action is intransitive. □

Example 3.2.1. Consider the action of $G = A_{15}$ on $X^{[5]}$. Then, $|X^{[5]}| = {}_{15}P_5 = \frac{15!}{10!}$ and by Lemma 3.2.1, $|H| = \frac{10!}{2}$. Now, by Theorem 1.1.2,

$$|Orb_G[x_1, x_2, x_3, x_4, x_5]| = \frac{|G|}{|H|} = \frac{15!/2}{10!/2} = \frac{15!}{10!} = |X^{[5]}|.$$

Therefore, by Definition 1.1.4, the action is transitive.

3.3 Primitivity of $\mathrm{A_n}$ on $\mathrm{X^{[2]}}$, $\mathrm{X^{[3]}}$, $\mathrm{X^{[4]}}$ and $\mathrm{X^{[r]}}$

3.3.1 Primitivity of $\mathrm{A_n}$ on $\mathrm{X^{[2]}}$

Proposition 3.3.1. The action of G on $X^{[2]}$ is imprimitive if and only if $n \geq 4$.

Proof. This action is transitive if and only if $n \geq 4$, by Theorem 3.2.1. Consider the non-trivial subset $Y = \{[1,2],[1,3],[1,4],\cdots,[1,n]\}$ of $X^{[2]}$ where $\bigcap\limits_{i=2}^{n} \{1,i\} = \{1\}$. Then $|Y| = n-1$ which divides $\left|X^{[2]}\right| = n(n-1)$. Now, if $g \in G$ such that 1 belongs to a 1-cycle of g, then g either fixes an element of Y or takes one element of Y to another so that $gY = Y$. Any other $g \in G$ moves an element of Y to an element not in Y so that $gY \cap Y = \emptyset$. This argument shows that Y is a block for the action and the conclusion follows, from Definition 1.1.8. $\square$

3.3.2 Primitivity of $\mathrm{A_n}$ on $\mathrm{X^{[3]}}$

Proposition 3.3.2. The action of G on $X^{[3]}$ is imprimitive if and only if $n \geq 5$.

Proof. The action is transitive if and only if $n \geq 5$, by Theorem 3.2.1. Consider the set $Y = \{[1,2,3],[1,2,4],[1,2,5],\cdots,[1,2,n]\}$ such that $\bigcap\limits_{i=3}^{n} \{1,2,i\} = \{1,2\}$. In this case $|Y| = n-2$ is a factor of $\left|X^{[3]}\right| = n(n-1)(n-2)$. Now, if $g \in G$ such that each of 1 and 2 belongs to a 1-cycle of g, then g either fixes an element of Y or takes one element of Y to another so that $gY = Y$. On the other hand, if either of 1 or 2 is in a k-cycle, where $k > 1$, of $g \in G$, then g moves an element of Y to an element not in Y so that $gY \cap Y = \emptyset$. Hence, Y is a non-trivial block for the action. Therefore, the action is imprimitive, by Definition 1.1.8. $\square$

3.3.3 Primitivity of $\mathrm{A_n}$ on $\mathrm{X^{[4]}}$

Proposition 3.3.3. The action of G on $X^{[4]}$ is imprimitive if and only if $n \geq 6$.

Proof. This action is transitive if and only if $n \geq 6$, by Theorem 3.2.1. Now, consider the set $Y = \{[1,2,3,4],[1,2,3,5],[1,2,3,6],\cdots,[1,2,3,n]\}$ with $\bigcap\limits_{i=4}^{n} \{1,2,3,i\} = \{1,2,3\}$. Clearly, $|Y| = n-3$ divides $\left|X^{[4]}\right| = n(n-1)(n-2)(n-3)$. Now, if $g \in G$ such that each of 1, 2, and 3 belongs to a 1-cycle of g, then g either fixes an element of Y or takes one element of Y to another so that $gY = Y$. Any other $g \in G$ moves an element of Y to an element not in Y so that $gY \cap Y = \emptyset$. Hence, Y is a non-trivial block for the action and the conclusion is direct, from Definition 1.1.8. $\square$

3.3.4 Primitivity of $\mathrm{A_n}$ on $\mathrm{X^{[r]}}$

Theorem 3.3.1. The action of G on $X^{[r]}$ is imprimitive if and only if $n \geq r+2$.

Proof. The action is transitive if and only if $n \geq r+2$, by Theorem 3.2.1. Consider the subset

$$\begin{aligned} Y &= \{[1,2,\cdots,r-1,r],[1,2,\cdots,r-1,r+1],[1,2,\cdots,r-1,r+2], \\ &\quad [1,2,\cdots,r-1,r+3],\cdots,[1,2,\cdots,r-1,n-1],[1,2,\cdots,r-1,n]\} \end{aligned}$$

of $X^{[r]}$ such that $\bigcap\limits_{i=r}^{n} \{1,2,\cdots,r-1,i\} = \{1,2,\cdots,r-1\}$. Then $|Y| = n-r+1$ which divides $\left|X^{[r]}\right| = n(n-1)(n-2)\cdots(n-r+1)$. Now, if $g \in G$ where each of $1,2,\cdots,r-1$ belongs to a 1-cycle of g, then g either fixes an element of Y or takes one element of Y to another so that $gY = Y$. However, if any of $1,2,\cdots,r-1$ belongs to a k-cycle, where $k > 1$, of $g \in G$, then g moves an element of Y to an element not in Y so that $gY \cap Y = \emptyset$. The argument shows that Y is a non-trivial block for the action. Hence the action is imprimitive, by Definition 1.1.8. $\square$

Example 3.3.1. Let $G = A_4$ act on $X^{[2]}$. The group G and the set $X^{[2]}$ in this case are as given in Subsubsection 3.2.1.2. Consider the subset $Y = \{[1,2],[1,3],[1,4]\}$ of $X^{[2]}$. Then, it can be shown that $gY = Y$ if $g = 1$, $g = (234)$ or $g = (243)$, and $gY \cap Y = \emptyset$ for any other $g \in G$. So, Y is a non-trivial block for the action. Therefore, by Definition 1.1.8, the action is imprimitive.

3.4 Ranks and Subdegrees of $\mathbf{A_n}$ on $\mathbf{X^{[2]}}$, $\mathbf{X^{[3]}}$, $\mathbf{X^{[4]}}$ and $\mathbf{X^{[r]}}$

3.4.1 Rank and Subdegrees of $\mathbf{A_n}$ on $\mathbf{X^{[2]}}$

From Theorem 3.2.1, this action is transitive if and only if $n \geq 4$. Throughout this subsection, by N it shall mean the set $\{1,2\}$.

3.4.1.1 Rank and Subdegrees of $\mathbf{A_4}$ on $\mathbf{X^{[2]}}$

Let G act on $X^{[2]}$. Then, $|X^{[2]}| = {}_4P_2 = 12$. From Lemma 3.2.1, $G_{[1,2]} = \{1\}$. Now, $G_{[1,2]}$ has orbits ($G-$suborbits) each of whose element has exactly 2, 1 or no element from N. These are

a) Suborbits whose respective elements contain both 1 and 2:

$\triangle_0 = Orb_{G_{[1,2]}}[1,2] = \{[1,2]\}$

$\triangle_1 = Orb_{G_{[1,2]}}[2,1] = \{[2,1]\}$

b) Suborbits whose respective elements contain exactly one element from N:

$\triangle_2 = Orb_{G_{[1,2]}}[1,3] = \{[1,3]\}$

$\triangle_3 = Orb_{G_{[1,2]}}[3,1] = \{[3,1]\}$

$\triangle_4 = Orb_{G_{[1,2]}}[1,4] = \{[1,4]\}$

$\triangle_5 = Orb_{G_{[1,2]}}[4,1] = \{[4,1]\}$

$\triangle_6 = Orb_{G_{[1,2]}}[2,3] = \{[2,3]\}$

$\triangle_7 = Orb_{G_{[1,2]}}[3,2] = \{[3,2]\}$

$\triangle_8 = Orb_{G_{[1,2]}}[2,4] = \{[2,4]\}$
$\triangle_9 = Orb_{G_{[1,2]}}[4,2] = \{[4,2]\}$
c) Suborbits whose respective elements contain neither 1 nor 2:
$\triangle_{10} = Orb_{G_{[1,2]}}[3,4] = \{[3,4]\}$
$\triangle_{11} = Orb_{G_{[1,2]}}[4,3] = \{[4,3]\}$
So, the action has 12 suborbits each of length 1. These are summarized in Table 3.1 below.

Table 3.1: Rank and Subdegrees of A_4 on $X^{[2]}$

Number x of Elements from N	Number of Suborbits Containing Exactly x Elements from N	Corresponding Subdegrees	Corresponding Number of Elements in $X^{[2]}$
2	${}_2C_2 \times {}_2P_2 = 2$	1	2
1	$({}_2C_1 \times {}_2P_1) \times 2 = 8$	1	8
0	$({}_2C_0 \times {}_2P_0) \times 2 = 2$	1	2
Total	12		$12 = \lvert X^{[2]}\rvert$

Thus, the rank of A_4 on $X^{[2]}$ is 12.

3.4.1.2 Rank and Subdegrees of $\mathbf{A_5}$ on $\mathbf{X^{[2]}}$

In this case $|X^{[2]}| = {}_5P_2 = 20$. From Lemma 3.2.1, $G_{[1,2]} \cong A_3$. Just like in the case in Subsubsection 3.4.1.1 above, $G_{[1,2]}$ has orbits each of whose every element has exactly 2, 1 or no element from N. These are summarized in Table 3.2 below.

Table 3.2: Rank and Subdegrees of A_5 on $X^{[2]}$

Number x of Elements from N	Number of Suborbits Containing Exactly x Elements from N	Corresponding Subdegrees	Corresponding Number of Elements in $X^{[2]}$
2	${}_2C_2 \times {}_2P_2 = 2$	1	2
1	${}_2C_1 \times {}_2P_1 = 4$	3	12
0	$({}_2C_0 \times {}_2P_0) \times 2 = 2$	3	6
Total	8		$20 = \lvert X^{[2]}\rvert$

So, the rank of A_5 on $X^{[2]}$ is 8.

3.4.1.3 Rank and Subdegrees of $\mathbf{A_6}$ on $\mathbf{X^{[2]}}$

In this case $|X^{[2]}| = {}_6P_2 = 30$. From Lemma 3.2.1, $G_{[1,2]} \cong A_4$. Now, $G_{[1,2]}$ has orbits each of whose every element has exactly 2, 1 or no element from N. These suborbits of G are summarized in Table 3.3 below.

Table 3.3: Rank and Subdegrees of A_6 on $X^{[2]}$

Number x of Elements from N	Number of Suborbits Containing Exactly x Elements from N	Coresponding Subdegrees	Coresponding Number of Elements in $X^{[2]}$
2	${}_2C_2 \times {}_2P_2 = 2$	1	2
1	${}_2C_1 \times {}_2P_1 = 4$	4	16
0	${}_2C_0 \times {}_2P_0 = 1$	12	12
Total	7		$30 = \lvert X^{[2]} \rvert$

So, the rank of A_6 on $X^{[2]}$ is 7.

3.4.1.4 Rank and Subdegrees of $\mathbf{A_n}$ on $\mathbf{X^{[2]}}$ for $\mathbf{n \geq 6}$

Proposition 3.4.1. The rank of G on $X^{[2]}$ is 7 if $n \geq 6$.

Proof. Let G act on $X^{[2]}$. Then $G_{[1,2]}$ has orbits each of whose every element has exactly 2, 1 or no element from N. There is only ${}_2C_2 = 1$ way of selecting two elements from N and the two can be arranged in the two positions in ${}_2P_2 = 2$ ways. So, there are ${}_2C_2 \times {}_2P_2 = 2$ suborbits each of whose element contains both elements from N. Also, there are ${}_2C_1 = 2$ ways of selecting an element from N, which can occupy any of the two positions in ${}_2P_1 = 2$ ways. Hence, there are ${}_2C_1 \times {}_2P_1 = 4$ suborbits each of whose every element has exactly one element from N. Similarly, there is only ${}_2C_0 \times {}_2P_0 = 1$ suborbit whose each element contains no element from N. So, G has 7 suborbits in total. $\square$

The seven suborbits discussed in Theorem 3.4.1 are

a) Suborbits whose respective elements contain both 1 and 2:

$\triangle_0 = Orb_{G_{[1,2]}}[1,2] = \{[1,2]\}$

$\triangle_1 = Orb_{G_{[1,2]}}[2,1] = \{[2,1]\}$

b) Suborbits each of whose every element contains exactly one element from N:

$\triangle_2 = Orb_{G_{[1,2]}}[1,3] = \{[1,3],[1,4],[1,5],\cdots,[1,n]\}$

$\triangle_3 = Orb_{G_{[1,2]}}[3,1] = \{[3,1],[4,1],[5,1],\cdots,[n,1]\}$

$\triangle_4 = Orb_{G_{[1,2]}}[2,3] = \{[2,3],[2,4],[2,5],\cdots,[2,n]\}$

$\triangle_5 = Orb_{G_{[1,2]}}[3,2] = \{[3,2],[4,2],[5,2],\cdots,[n,2]\}$

c) Suborbit whose each element contains neither 1 nor 2:

$\triangle_6 = Orb_{G_{[1,2]}}[3,4] = \{[3,4],[3,5],\cdots,[3,n],[4,3],\cdots,[4,n],\cdots,[n,n-1]\}$

The subdegrees and corresponding number of suborbits of A_n on $X^{[2]}$ for $n \geq 6$ are summarized in Table 3.4 below.

Table 3.4: Rank and Subdegrees of A_n on $X^{[2]}$ for $n \geq 6$

Number of Elements from N	Corresponding Subdegrees	Corresponding Number of Suborbits
2	1	${}_2C_2 \times {}_2P_2 = 2$
1	$(n-2)$	${}_2C_1 \times {}_2P_1 = 4$
0	$(n-2)(n-3)$	${}_2C_0 \times {}_2P_0 = 1$
Total		7

3.4.2 Rank and Subdegrees of $\mathbf{A_n}$ on $\mathbf{X^{[3]}}$

From Theorem 3.2.1, this action is transitive if and only if $n \geq 5$. Throughout this subsection, by N it shall mean the set $\{1, 2, 3\}$.

3.4.2.1 Rank and Subdegrees of $\mathbf{A_5}$ on $\mathbf{X^{[3]}}$

Let G act on $X^{[3]}$. In this case $|X^{[3]}| = {}_5P_3 = 60$. From Lemma 3.2.1, $G_{[1,2,3]} = \{1\}$ and has orbits each of whose element has exactly 3, 2, or 1 element from N. Each of these suborbits of G has length 1 and they are 60 in total. The suborbits are summarized in Table 3.5 below.

Table 3.5: Rank and Subdegrees of A_5 on $X^{[3]}$

Number x of Elements from N	Number of Suborbits Containing Exactly x Elements from N	Corresponding Subdegrees	Corresponding Number of Elements in $X^{[3]}$
3	${}_3C_3 \times {}_3P_3 = 6$	1	6
2	$({}_3C_2 \times {}_3P_2) \times 2 = 36$	1	36
1	$({}_3C_1 \times {}_3P_1) \times 2 = 18$	1	18
Total	60		$60 = \lvert X^{[3]} \rvert$

So, the rank of A_5 on $X^{[3]}$ is 60.

3.4.2.2 Rank and Subdegrees of $\mathbf{A_6}$ on $\mathbf{X^{[3]}}$

If G acts on $X^{[3]}$, then $|X^{[3]}| = {}_6P_3 = 120$. From Lemma 3.2.1, $G_{[1,2,3]} \cong A_3$ and $G_{[1,2,3]}$ has orbits each of whose every element contains exactly 3, 2, 1 or no element from N. These suborbits of G are summarized in Table 3.6 below.

Table 3.6: Rank and Subdegrees of A_6 on $X^{[3]}$

Number x of Elements from N	Number of Suborbits Containing Exactly x Elements from N	Corresponding Subdegrees	Corresponding Number of Elements $X^{[3]}$
3	${}_3C_3 \times {}_3P_3 = 6$	1	6
2	${}_3C_2 \times {}_3P_2 = 18$	3	54
1	$({}_3C_1 \times {}_3P_1) \times 2 = 18$	3	54
0	$({}_3C_0 \times {}_3P_0) \times 2 = 2$	3	6
Total	44		$120 = \|X^{[3]}\|$

So, the rank of A_6 on $X^{[3]}$ is 44.

3.4.2.3 Rank and Subdegrees of $\mathbf{A_7}$ on $\mathbf{X^{[3]}}$

In this case $|X^{[3]}| = {}_7P_3 = 210$. From Lemma 3.2.1, $G_{[1,2,3]} \cong A_4$ and $G_{[1,2,3]}$ has orbits each of whose every element has exactly 3, 2, 1 or no element from N. These suborbits of G are summarized in Table 3.7 below.

Table 3.7: Rank and Subdegrees of A_7 on $X^{[3]}$

Number x of Elements from N	Number of Suborbits Containing Exactly x Elements from N	Corresponding Subdegrees	Corresponding Number of Elements in $X^{[3]}$
3	${}_3C_3 \times {}_3P_3 = 6$	1	6
2	${}_3C_2 \times {}_3P_2 = 18$	4	72
1	${}_3C_1 \times {}_3P_1 = 9$	12	108
0	$({}_3C_0 \times {}_3P_0) \times 2 = 2$	12	24
Total	35		$210 = \|X^{[3]}\|$

So, the rank of A_7 on $X^{[3]}$ is 35.

3.4.2.4 Rank and Subdegrees of $\mathbf{A_8}$ on $\mathbf{X^{[3]}}$

Let G act on $X^{[3]}$. Then, $|X^{[3]}| = {}_8P_3 = 336$. From Lemma 3.2.1, $G_{[1,2,3]} \cong A_5$ and $G_{[1,2,3]}$ has orbits each of whose every element has exactly 3, 2, 1, or no element from N. These suborbits of G are summarized in Table 3.8 below.

Table 3.8: Rank and Subdegrees of A_8 on $X^{[3]}$

Number x of Elements from N	Number of Suborbits Containing Exactly x Elements from N	Corresponding Subdegrees	Corresponding Number of Elements in $X^{[3]}$
3	${}_3C_3 \times {}_3P_3 = 6$	1	6
2	${}_3C_2 \times {}_3P_2 = 18$	5	90
1	${}_3C_1 \times {}_3P_1 = 9$	20	180
0	${}_3C_0 \times {}_3P_0 = 1$	60	60
Total	34		$336 = \|X^{[3]}\|$

So, the rank of A_8 on $X^{[3]}$ is 34.

3.4.2.5 Rank and Subdegrees of $\mathbf{A_n}$ on $\mathbf{X^{[3]}}$ for $\mathbf{n \geq 8}$

Proposition 3.4.2. The rank of G on $X^{[3]}$ is 34 for all $n \geq 8$.

Proof. Suppose G acts on $X^{[3]}$. Then $G_{[1,2,3]}$ has orbits each of whose element has exactly 3, 2, 1, or no element from N. An argument similar to the one in the proof of Theorem 3.4.1 shows that G has ${}_3C_3 \times {}_3P_3 = 6$ suborbits each of whose element has exactly 3 elements from N, ${}_3C_2 \times {}_3P_2 = 18$ suborbits each of whose every element has exactly 2 elements from N, ${}_3C_1 \times {}_3P_1 = 9$ suborbits each of whose every element has exactly 1 element from N and ${}_3C_0 \times {}_3P_0 = 1$ suborbit whose every element has no element from N. Therefore, the rank of G on $X^{[3]}$ is 34. $\square$

The 34 suborbits $\triangle_0, \triangle_1, \triangle_2, \cdots, \triangle_{33}$ discussed in Theorem 3.4.2 are listed in Appendix A. The subdegrees and corresponding number of suborbits of A_n on $X^{[3]}$ for $n \geq 8$ are summarized in the Table 3.9 below.

Table 3.9: Rank and Subdegrees of A_n on $X^{[3]}$ for $n \geq 8$

Number of Elements from N	Corresponding Subdegrees	Corresponding Number of Suborbits
3	1	${}_3C_3 \times {}_3P_3 = 6$
2	$(n-3)$	${}_3C_2 \times {}_3P_2 = 18$
1	$(n-3)(n-4)$	${}_3C_1 \times {}_3P_1 = 9$
0	$(n-3)(n-4)(n-5)$	${}_3C_0 \times {}_3P_0 = 1$
Total		34

3.4.3 Rank and Subdegrees of $\mathbf{A_n}$ on $\mathbf{X^{[4]}}$

From Theorem 3.2.1, this action is transitive if and only if $n \geq 6$. Throughout this subsection, by N it shall mean the set $\{1, 2, 3, 4\}$.

3.4.3.1 Rank and Subdegrees of $\mathbf{A_6}$ on $\mathbf{X^{[4]}}$

Let G act on $X^{[4]}$. Then $|X^{[4]}| = {}_6P_4 = 360$. From Lemma 3.2.1, $G_{[1,2,3,4]} = \{1\}$, and has orbits each of whose element has exactly 4, 3, or 2 elements from N. Each of these suborbits of G has length 1 and they are 360 in number. These suborbits are summarized in Table 3.10 below.

Table 3.10: Rank and Subdegrees of A_6 on $X^{[4]}$

Number x of Elements from N	Number of Suborbits Containing Exactly x Elements from N	Corresponding Subdegrees	Corresponding Number of Elements in $X^{[4]}$
4	${}_4C_4 \times {}_4P_4 = 24$	1	24
3	$({}_4C_3 \times {}_4P_3) \times 2 = 192$	1	192
2	$({}_4C_2 \times {}_4P_2) \times 2 = 144$	1	144
Total	360		$360 = |X^{[4]}|$

So, the rank of A_6 on $X^{[4]}$ is 360.

3.4.3.2 Rank and Subdegrees of $\mathbf{A_7}$ on $\mathbf{X^{[4]}}$

Suppose G acts on $X^{[4]}$. Then, $|X^{[4]}| = {}_7P_4 = 840$. From Lemma 3.2.1, $G_{[1,2,3,4]} \cong A_3$ and $G_{[1,2,3,4]}$ has orbits each of whose every element has exactly 4, 3, 2, or 1 element from N. These suborbits of G are summarized in Table 3.11 below.

Table 3.11: Rank and Subdegrees of A_7 on $X^{[4]}$

Number x of Elements from N	Number of Suborbits Containing Exactly x Elements from N	Corresponding Subdegrees	Corresponding Number of Elements in $X^{[4]}$
4	${}_4C_4 \times {}_4P_4 = 24$	1	24
3	${}_4C_3 \times {}_4P_3 = 96$	3	288
2	$({}_4C_2 \times {}_4P_2) \times 2 = 144$	3	432
1	$({}_4C_1 \times {}_4P_1) \times 2 = 32$	3	96
Total	296		$840 = |X^{[4]}|$

So, the rank of A_7 on $X^{[4]}$ is 296.

3.4.3.3 Rank and Subdegrees of $\mathbf{A_8}$ on $\mathbf{X^{[4]}}$

In this case $|X^{[4]}| = {}_8P_4 = 1680$. From Lemma 3.2.1, $G_{[1,2,3,4]} \cong A_4$. Moreover, $G_{[1,2,3,4]}$ has orbits each of whose every element contains exactly 4, 3, 2, 1, or no element from N. These suborbits of G are summarized in Table 3.12 below.

Table 3.12: Rank and Subdegrees of A_8 on $X^{[4]}$

Number x of Elements from N	Number of Suborbits Containing Exactly x Elements from N	Corresponding Subdegrees	Corresponding Number of Elements in $X^{[4]}$
4	${}_4C_4 \times {}_4P_4 = 24$	1	24
3	${}_4C_3 \times {}_4P_3 = 96$	4	384
2	${}_4C_2 \times {}_4P_2 = 72$	12	864
1	$({}_4C_1 \times {}_4P_1) \times 2 = 32$	12	384
0	$({}_4C_0 \times {}_4P_0) \times 2 = 2$	12	24
Total	226		$1680 = \lvert X^{[4]}\rvert$

So, the rank of A_8 on $X^{[4]}$ is 226.

3.4.3.4 Rank and Subdegrees of $\mathbf{A_9}$ on $\mathbf{X^{[4]}}$

In this case $|X^{[4]}| = {}_9P_4 = 3024$. From Lemma 3.2.1, $G_{[1,2,3,4]} \cong A_5$. Further, $G_{[1,2,3,4]}$ has orbits each of whose every element has exactly 4, 3, 2, 1, or no element from N. These suborbits of G are summarized in Table 3.13 below.

Table 3.13: Rank and Subdegrees of A_9 on $X^{[4]}$

Number x of Elements from N	Number of Suborbits Containing Exactly x Elements from N	Corresponding Subdegrees	Corresponding Number of Elements in $X^{[4]}$
4	${}_4C_4 \times {}_4P_4 = 24$	1	24
3	${}_4C_3 \times {}_4P_3 = 96$	5	480
2	${}_4C_2 \times {}_4P_2 = 72$	20	1440
1	${}_4C_1 \times {}_4P_1 = 16$	60	960
0	$({}_4C_0 \times {}_4P_0) \times 2 = 2$	60	120
Total	210		$3024 = \lvert X^{[4]}\rvert$

So, the rank of A_9 on $X^{[4]}$ is 210.

3.4.3.5 Rank and Subdegrees of $\mathbf{A_{10}}$ on $\mathbf{X^{[4]}}$

Let G act on $X^{[4]}$. Then, $|X^{[4]}| = {}_{10}P_4 = 5040$. From Lemma 3.2.1, $G_{[1,2,3,4]} \cong A_6$ and $G_{[1,2,3,4]}$ has orbits each of whose every element has exactly 4, 3, 2, 1, or no element from N. These suborbits of G are summarized in Table 3.14 below.

Table 3.14: Rank and Subdegrees of A_{10} on $X^{[4]}$

Number x of Elements from N	Number of Suborbits Containing Exactly x Elements from N	Corresponding Subdegrees	Corresponding Number of Elements in $X^{[4]}$
4	${}_4C_4 \times {}_4P_4 = 24$	1	24
3	${}_4C_3 \times {}_4P_3 = 96$	6	576
2	${}_4C_2 \times {}_4P_2 = 72$	30	2160
1	${}_4C_1 \times {}_4P_1 = 16$	120	1920
0	${}_4C_0 \times {}_4P_0 = 1$	360	360
Total	209		$5040 = \lvert X^{[4]}\rvert$

So, the rank of A_{10} on $X^{[4]}$ is 209.

3.4.3.6 Rank and Subdegrees of $\mathbf{A_n}$ on $\mathbf{X^{[4]}}$ for $\mathbf{n \geq 10}$

Proposition 3.4.3. The rank of G on $X^{[4]}$ is 209 for all $n \geq 10$.

Proof. It is analogous to the proofs of Propositions 3.4.1 and 3.4.2 above. □

The 209 suborbits $\triangle_0, \triangle_1, \cdots, \triangle_{208}$ discussed in Theorem 3.4.3 are as listed in Appendix B. The subdegrees and corresponding number of suborbits of A_n on $X^{[4]}$ for $n \geq 10$ are summarized in Table 3.15 below.

Table 3.15: Rank and Subdegrees of A_n on $X^{[4]}$ for $n \geq 10$

Number of Elements from N	Corresponding Subdegrees	Corresponding Number of Suborbits
4	1	${}_4C_4 \times {}_4P_4 = 24$
3	$(n-4)$	${}_4C_3 \times {}_4P_3 = 96$
2	$(n-4)(n-5)$	${}_4C_2 \times {}_4P_2 = 72$
1	$(n-4)(n-5)(n-6)$	${}_4C_1 \times {}_4P_1 = 16$
0	$(n-4)(n-5)(n-6)(n-7)$	${}_4C_0 \times {}_4P_0 = 1$
Total		209

3.4.4 Rank and Subdegrees of $\mathbf{A_n}$ on $\mathbf{X^{[r]}}$

Let G act on $X^{[r]}$ and let $N = \{1, 2, \cdots, r\}$. From the findings in Subsections 3.4.1 through 3.4.3, above, it is quite clear that if $n \geq 2(r+1)$, G has suborbits each of whose every element has exactly $r - i$ ($i = 0, 1, 2, \cdots, r$) elements from N. The subdegrees and corresponding number of suborbits of the action are obtained by generalizing the results in, respectively, the second and third columns of Tables 3.4, 3.9 and 3.15 above. This is as shown in Table 3.16 below.

Table 3.16: Rank and Subdegrees of A_n on $X^{[r]}$ for $n \geq 2(r+1)$

Number of Elements from N	Corresponding Subdegrees	Corresponding Number of Suborbits
r	1	${}_rC_r \times_r P_r$
$(r-1)$	$(n-r)$	${}_rC_{r-1} \times_r P_{r-1}$
$(r-2)$	$(n-r)(n-r-1)$	${}_rC_{r-2} \times_r P_{r-2}$
$(r-3)$	$(n-r)(n-r-1)(n-r-2)$	${}_rC_{r-3} \times_r P_{r-3}$
$(r-4)$	$(n-r)(n-r-1)(n-r-2)(n-r-3)$	${}_rC_{r-4} \times_r P_{r-4}$
$\vdots$	$\vdots$	$\vdots$
$(r-i)$	$(n-r)(n-r-1)\cdots(n-r-i+1)$	${}_rC_{r-i} \times_r P_{r-i}$
$\vdots$	$\vdots$	$\vdots$
3	$(n-r)(n-r-1)\cdots(n-2r+4)$	${}_rC_3 \times_r P_3$
2	$(n-r)(n-r-1)\cdots(n-2r+3)$	${}_rC_2 \times_r P_2$
1	$(n-r)(n-r-1)\cdots(n-2r+2)$	${}_rC_1 \times_r P_1$
0	$(n-r)(n-r-1)\cdots(n-2r+1)$	${}_rC_0 \times_r P_0$
Total		$\sum_{i=0}^{r} ({}_rC_{r-i} \times_r P_{r-i})$

Lemma 3.4.1. Let G act on $X^{[r]}$ and let $n \geq 2(r+1)$. Suppose a suborbit $\triangle_i$ of G has exactly $r-i$ ($i = 0, 1, 2, \cdots, r$) elements from $N = \{1, 2, \cdots, r\}$. Then adding an extra element to the set X increases $|\triangle_i|$ by

$$i(n-r)(n-r-1)(n-r-2)\cdots(n-r-i+3)(n-r-i+2)$$

units; however, this addition does not affect the rank of G.

Proof. From the second column of Table 3.16,

$$|\triangle_i| = (n-r)(n-r-1)(n-r-2)\cdots(n-r-i+2)(n-r-i+1).$$

If an extra element is added to X, the new value of $|\triangle_i|$ is obtained by replacing n with $n+1$, which equals $(n-r+1)(n-r)(n-r-1)\cdots(n-r-i+3)(n-r-i+2)$. So, the number of units by which the suborbit length changes is

$$\begin{aligned}
&(n-r+1)(n-r)(n-r-1)\cdots(n-r-i+3)(n-r-i+2) \\
&\qquad -(n-r)(n-r-1)\cdots(n-r-i+2)(n-r-i+1) \\
&\qquad = [(n-r+1)-(n-r-i+1)] \\
&\qquad \times(n-r)(n-r-1)\cdots(n-r-i+3)(n-r-i+2) \\
&= i(n-r)(n-r-1)\cdots(n-r-i+3)(n-r-i+2).
\end{aligned}$$

Now, the number of suborbits $\triangle_i$, which is the corresponding entry in the third column of

Table 3.16, is given purely in terms of the non-negative integers r and i. It is clear that these integers are unaffected by increasing the number of elements of X. This in turn implies that increasing $|X|$ does not change the number of suborbits $\triangle_i$. Accordingly, the rank of G, which is simply the sum of entries in the said column, is not affected by adding an extra element to X. $\square$

Theorem 3.4.1. The rank of G on $X^{[r]}$ is $(r!)^2 \sum_{i=0}^{r} \frac{1}{(i!)^2(r-i)!}$ for all $n \geq 2(r+1)$.

Proof. If $n = 2(r+1)$, the sum of the entries in the third column of Table 3.16 gives the desired result; in other words, the rank of G is

$$\begin{aligned}\sum_{i=0}^{r} ({}_rC_{r-i} \times {}_rP_{r-i}) &= \sum_{i=0}^{r} \left(\frac{r!}{i!(r-i)!} \times \frac{r!}{i!} \right) \\ &= (r!)^2 \sum_{i=0}^{r} \frac{1}{(i!)^2\,(r-i)!}.\end{aligned}$$

Thus, the given statement is true for $n = 2(r+1)$. Now, suppose the statement is true for $n = 2(r+1)+k$ where $k \in \mathbb{Z}^+$. Now, for $n = 2(r+1)+(k+1)$, add an extra element to the set $\{1, 2, \cdots, 2r, 2r+1, 2(r+1), \cdots, 2(r+1)+k\}$. By Lemma 3.4.1, the extra element just changes the length of each suborbit $\triangle_i$ ($i = 0, 1, 2, \cdots, r$) whose every element contains $r-i$ elements from the set $\{1, 2, \cdots, r\}$, by

$$i(n-r)(n-r-1)\cdots(n-r-i+3)(n-r-i+2)$$

units, but this increment has no effect on the number of the suborbits $\triangle_i$. As a result, if $n = 2(r+1)+(k+1)$, the rank is the same as that when $n = 2(r+1)+k$. So, the statement holds for $n = 2(r+1)+(k+1)$ whenever it holds for $n = 2(r+1)+k$. Therefore, by the principle of mathematical induction, the statement is true for all $n \geq 2(r+1)$. $\square$

Example 3.4.1. The group A_n $(n \geq 10)$ acts on $X^{[4]}$ with rank

$$\begin{aligned}\kappa &= (4!)^2 \left[\frac{1}{(0!)^2 4!} + \frac{1}{(1!)^2 3!} + \frac{1}{(2!)^2 2!} + \frac{1}{(3!)^2 1!} + \frac{1}{(4!)^2 0!} \right] \\ &= 576 \left[\frac{1}{24} + \frac{1}{6} + \frac{1}{8} + \frac{1}{36} + \frac{1}{576} \right] \\ &= 576 \times \frac{209}{576} \\ &= 209.\end{aligned}$$

Also, $A_n\,(n \geq 12)$ acts on $X^{[5]}$ with rank

$$\begin{aligned}
\mu &= (5!)^2\left[\frac{1}{(0!)^2 5!}+\frac{1}{(1!)^2 4!}+\frac{1}{(2!)^2 3!}+\frac{1}{(3!)^2 2!}+\frac{1}{(4!)^2 1!}+\frac{1}{(5!)^2 0!}\right]\\
&= 14400\left[\frac{1}{120}+\frac{1}{24}+\frac{1}{24}+\frac{1}{72}+\frac{1}{576}+\frac{1}{14400}\right]\\
&= 14400\times\frac{773}{7200}\\
&= 1546.
\end{aligned}$$

In a similar manner, it can be shown that the rank of $A_n\,(n \geq 14)$ on $X^{[6]}$ is 13327 while that of $A_n\,(n \geq 16)$ on $X^{[7]}$ is 130922. In fact, the Python computer programme in Appendix C can be used to compute the rank of A_n on $X^{[r]}$ for all $n \geq 2(r+1)$ whenever $r \leq 166$.

3.5 Pairing of Suborbits of $\mathbf{A_n}$ on $\mathbf{X^{[r]}}$

Theorem 3.5.1. Let $\triangle$ be an orbit of $G_{[1,2,\cdots,r]}$ on $X^{[r]}$. Suppose the ordered $r-$element subsets $[x_1, x_2, \cdots, x_r]$ and $[y_1, y_2, \cdots, y_r]$ are in $\triangle$. Then $\triangle$ is self-paired if and only if $\exists$ some permutations $g_i = \begin{pmatrix} 1 & 2 & \cdots & r & \cdots & k \\ y_1 & y_2 & \cdots & y_r & \cdots & y_k \end{pmatrix}$ and $g_j = \begin{pmatrix} 1 & 2 & \cdots & r & \cdots & k \\ x_1 & x_2 & \cdots & x_r & \cdots & x_k \end{pmatrix}$ in G, with $r \leq k \leq n$, that are inverses of each other. In case $[x_1, x_2, \cdots, x_r] = [y_1, y_2, \cdots, y_r]$, then $g_i = g_j = g$ such that $g = \begin{pmatrix} 1 & 2 & \cdots & r & \cdots & k \\ x_1 & x_2 & \cdots & x_r & \cdots & x_k \end{pmatrix}$ is self-inverse.

Proof. Suppose $\triangle$ is self-paired. Then, by Definition 1.1.7 there exist $g_i, g_j \in G$ such that

$$g_i[x_1, x_2, \cdots, x_r] = [1, 2, \cdots, r];\; g_i[1, 2, \cdots, r] = [y_1, y_2, \cdots, y_r]$$

and

$$g_j[y_1, y_2, \cdots, y_r] = [1, 2, \cdots, r];\; g_j[1, 2, \ldots, r] = [x_1, x_2, \cdots, x_r].$$

By the definition of the action,

$$g_i(x_1) = 1, g_i(x_2) = 2, \cdots, g_i(x_r) = r;\; g_i(1) = y_1, g_i(2) = y_2, \cdots, g_i(r) = y_r$$

and

$$g_j(y_1) = 1, g_j(y_2) = 2, \cdots, g_j(y_r) = r;\; g_j(1) = x_1, g_j(2) = x_2, \cdots, g_j(r) = x_r.$$

This argument implies that

$$(g_ig_j)(1) = 1, (g_ig_j)(2) = 2, \cdots, (g_ig_j)(r) = r$$

and

$$(g_jg_i)(1) = 1, (g_jg_i)(2) = 2, \cdots, (g_jg_i)(r) = r,$$

so that the permutations g_i and g_j are inverses of each other.
Conversely, suppose there exist even permutations $g_i = \begin{pmatrix} 1 & 2 & \cdots & r & \cdots & k \\ y_1 & y_2 & \cdots & y_r & \cdots & y_k \end{pmatrix}$ and $g_j = \begin{pmatrix} 1 & 2 & \cdots & r & \cdots & k \\ x_1 & x_2 & \cdots & x_r & \cdots & x_k \end{pmatrix}$ that are inverses of each other. Then $g_i[x_1, x_2, \cdots, x_r] = [1, 2, \cdots, r]$ and $g_i[1, 2, \cdots, r] = [y_1, y_2, \cdots, y_r]$. On the other hand, $g_j[y_1, y_2, \cdots, y_r] = [1, 2, \cdots, r]$ and $g_j[1, 2, \cdots, r] = [x_1, x_2, \cdots, x_r]$. Hence, $\triangle$ is self-paired, by Definition 1.1.7. Now, suppose $[x_1, x_2, \cdots, x_r] = [y_1, y_2, \cdots, y_r]$. Then, clearly, $g_i = g_j = g$ and it is trivially self-inverse. □

Theorem 3.5.2. Suppose G acts on $X^{[r]}$ and suppose $\triangle_i$ and $\triangle_j$ are orbits of $G_{[1,2,\cdots,r]}$. Let $[x_1, x_2, \cdots, x_r] \in \triangle_i$ where $x_k \in \{1, 2, \cdots, n\}$ $\forall k = 1, 2, \cdots, r$. Then $\triangle_i$ is paired with $\triangle_j$ if and only if there is an element $[y_1, y_2, \cdots, y_r] \in \triangle_j$ with $y_t \in \{1, 2, \cdots, n\}$ $\forall t = 1, 2, \cdots, r$, and some even permutations $g_i = \begin{pmatrix} 1 & 2 & \cdots & r & \cdots & k \\ y_1 & y_2 & \cdots & y_r & \cdots & y_k \end{pmatrix}$ and $g_j = \begin{pmatrix} 1 & 2 & \cdots & r & \cdots & k \\ x_1 & x_2 & \cdots & x_r & \cdots & x_k \end{pmatrix}$, $r \leq k \leq n$, that are inverses of each other.

Proof. Suppose $\triangle_i$ is paired with $\triangle_j$ and $[x_1, x_2, \cdots, x_r] \in \triangle_i$. Then , by Definition 1.1.7, there exists $[y_1, y_2, \cdots, y_r] \in \triangle_j$ and $g_i, g_j \in G$ such that

$$g_i[x_1, x_2, \cdots, x_r] = [1, 2, \cdots, r];\ g_i[1, 2, \cdots, r] = [y_1, y_2, \cdots, y_r]$$

and

$$g_j[y_1, y_2, \cdots, y_r] = [1, 2, \cdots, r];\ g_j[1, 2, \cdots, r] = [x_1, x_2, \cdots, x_r].$$

By the definition of the action,

$$g_i(x_1) = 1, g_i(x_2) = 2, \cdots, g_i(x_r) = r;\ g_i(1) = y_1, g_i(2) = y_2, \cdots, g_i(r) = y_r$$

and

$$g_j(y_1) = 1, g_j(y_2) = 2, \cdots, g_j(y_r) = r;\ g_j(1) = x_1, g_j(2) = x_2, \cdots, g_j(r) = x_r.$$

This implies that

$$(g_ig_j)(1) = 1, (g_ig_j)(2) = 2, \cdots, (g_ig_j)(r) = r$$

and

$$(g_jg_i)(1) = 1, (g_jg_i)(2) = 2, \cdots, (g_jg_i)(r) = r,$$

so that the permutations g_i and g_j are inverses of each other.

Conversely, suppose there exist even permutations $g_i = \begin{pmatrix} 1 & 2 & \cdots & r & \cdots & k \\ y_1 & y_2 & \cdots & y_r & \cdots & y_k \end{pmatrix}$ and $g_j = \begin{pmatrix} 1 & 2 & \cdots & r & \cdots & k \\ x_1 & x_2 & \cdots & x_r & \cdots & x_k \end{pmatrix}$ that are inverses of each other. Then $g_i[x_1, x_2, \cdots, x_r] = [1, 2, \cdots, r]$ and $g_i[1, 2, \cdots, r] = [y_1, y_2, \cdots, y_r]$. On the other hand, $g_j[y_1, y_2, \cdots, y_r] = [1, 2, \cdots, r]$ and $g_j[1, 2, \cdots, r] = [x_1, x_2, \cdots, x_r]$. Hence, $\triangle_i$ and $\triangle_j$ are paired, by Definition 1.1.7. $\square$

Example 3.5.1. Consider the action of G on $X^{[2]}$ for $n \geq 6$ (see Subsection 3.4.1). Since $|G|$ is even, then by Theorem 1.1.4, $G_{[1,2]}$ has at least one non-trivial orbit which is self-paired. Now, consider $[2, 1] \in \triangle_1$ and the self-inverse even permutation $g = (12)(34) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$. In this case, g takes $[2, 1]$ to $[1, 2]$ and vice versa. So, by Definition 1.1.7, $\triangle_1^* = \triangle_1$, that is $\triangle_1$ is self-paired. A similar argument shows that $\triangle_2^* = \triangle_2$, $\triangle_5^* = \triangle_5$ and $\triangle_6^* = \triangle_6$. On the other hand, $[3, 1] \in \triangle_3$, and if $g_1 = (123) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$, then $g_1[3, 1] = [1, 2]$ and $g_1[1, 2] = [2, 3] \in \triangle_4$. Further, if $g_2 = (132) = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$, then $g_2[2, 3] = [1, 2]$ and $g_2[1, 2] = [3, 1]$. In this case, g_1 and g_2 are inverses of each other and by Definition 1.1.7, $\triangle_3^* = \triangle_4$, that is $\triangle_3$ is paired with $\triangle_4$.

Example 3.5.2. From Appendix A, $\triangle_{17}$, $\triangle_{21}$ and $\triangle_{23}$ are suborbits associated with the action of G on $X^{[3]}$ for $n \geq 8$. Take the even permutation $g_1 = (1654)(23) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 3 & 2 & 1 & 4 & 5 \end{pmatrix}$. Then $g_1[4, 3, 2] = [1, 2, 3] \in g_1\triangle_{23}$ and $g_1[1, 2, 3] = [6, 3, 2] \in \triangle_{23}$. Similarly, consider the permutation $g_2 = (1456)(23) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 3 & 2 & 5 & 6 & 1 \end{pmatrix}$. Then $g_2[6, 3, 2] = [1, 2, 3]$ and $g_2[1, 2, 3] = [4, 3, 2]$. In this case g_1 and g_2 are inverses of each other. Alternatively, the self-inverse permutation $g = (14)(23) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$ takes $[4, 3, 2]$ to $[1, 2, 3]$ and vice versa. So, by Definition 1.1.7, $\triangle_{23}$ is self-paired. On the other hand, consider the even permutation $g_3 = (1324)(56) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 2 & 1 & 6 & 5 \end{pmatrix}$. Then $g_3[4, 3, 1] = [1, 2, 3] \in g_3\triangle_{17}$ and $g_3[1, 2, 3] = [3, 4, 2] \in \triangle_{21}$. Also, let $g_4 = (1423)(56) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 3 & 1 & 2 & 6 & 5 \end{pmatrix}$. Then $g_4[3, 4, 2] = [1, 2, 3]$ and $g_4[1, 2, 3] = [4, 3, 1]$. In this case g_3 and g_4 are inverses of each other, and by Definition 1.1.7, $\triangle_{17}$ is paired with $\triangle_{21}$. In fact, the 13 suborbits $\triangle_i\,(i = 1, 2, 5, 6, 8, 13, 15, 22, 23, 24, 28, 32$ and $33)$ of the action are self-paired while the suborbits $\triangle_j\,(j = 3, 7, 9, 10, 11, 16, 17, 25, 26, 29)$ are paired, respectively, with the suborbits $\triangle_k\,(k = 4, 12, 14, 18, 20, 19, 21, 27, 30, 31)$.

Example 3.5.3. Similarly, consider the suborbits of the action of G on $X^{[4]}$ for $n \geq 10$ (see these suborbits in Appendix B). The suborbits $\triangle_i$ $(i = 1, 2, 5, 6, 7, 14, 16, 21, 23, 24, 28, 30, 40, 49, 51, 55, 63, 76, 77, 82, 89, 114, 115, 116, 119, 120, 121, 134, 135, 148, 149, 160, 161, 176, 177, 190, 191, 192, 197, 202, 207, 208)$ are self-paired. On the other hand, the suborbits $\triangle_j$ $(j = 3, 8, 9, 10, 11, 15, 17, 25, 26, 27, 29, 31, 32, 33, 34, 35, 37, 38, 39, 41, 42, 43, 44, 45, 46, 47, 52, 53, 57, 58, 59, 64, 65, 66, 67, 68, 69, 70, 71, 83, 90, 91, 92, 93, 94, 95, 117, 122, 123, 124, 125, 126, 127, 128, 129, 130, 131, 136, 137, 138, 139, 140, 141, 142, 143, 150, 151, 152, 153, 154, 155, 164, 165, 166, 167, 178, 179, 193, 194, 195, 198, 199, 203)$ are paired, respectively, with the suborbits $\triangle_k$ $(k = 4, 12, 18, 13, 19, 20, 22, 48, 72, 74, 50, 54, 36, 60, 78, 84, 56, 80, 86, 62, 96, 98, 102, 108, 104, 110, 73, 75, 61, 79, 85, 81, 87, 97, 99, 103, 109, 105, 111, 88, 100, 101, 106, 112, 107, 113, 118, 132, 133, 144, 145, 180, 181, 168, 169, 156, 157, 146, 147, 182, 183, 170, 171, 158, 159, 162, 163, 172, 173, 184, 185, 174, 175, 186, 187, 188, 189, 196, 200, 204, 201, 205, 206)$.

Lemma 3.5.1. Let the cycle type of $g \in G$ be $(\alpha_1, \alpha_2, \cdots, \alpha_n)$. If $\alpha_1 \geq r$, then the number of elements in $X^{[r]}$ fixed by g is given by

$$|fix(g)| = r!\binom{\alpha_1}{r}.$$

Proof. Let $[x_1, x_2, \cdots, x_r] \in X^{[r]}$. Then $g \in G$ fixes $[x_1, x_2, \cdots, x_r]$ if and only if each of the elements $x_1, x_2, \cdots, x_r$ comes from a 1-cycle in g (see Proof of Lemma 3.2.1). From the set of α_1 elements of X that are fixed by g, the total number of ordered r-element subsets that can be formed is

$$_{\alpha_1}P_r = r!\binom{\alpha_1}{r},$$

and the conclusion is clear. □

Theorem 3.5.3. Let G act on $X^{[r]}$ and suppose $g \in G$ has cycle type $(\alpha_1, \alpha_2, \cdots, \alpha_n)$. Then the number of self-paired suborbits of G is given by

$$\pi = \frac{2r!}{n!}\sum_{g\in G}\binom{\alpha_1 + 2\alpha_2}{r}. \qquad (3.5.1)$$

Proof. The number of 1-cycles in g^2 is $(\alpha_1 + 2\alpha_2)$, by Theorem 1.1.13. By Lemma 3.5.1, the number of elements in $X^{[r]}$ fixed by g^2 is given by

$$|fix(g^2)| = r!\binom{\alpha_1 + 2\alpha_2}{r}.$$

Now, by Theorem 1.1.5, the number of self-paired suborbits of G on $X^{[r]}$ is given by

$$\begin{aligned}\frac{1}{|G|}\sum_{g\in G}|fix(g^2)| &= \frac{1}{n!/2}\sum_{g\in G} r!\binom{\alpha_1+2\alpha_2}{r} \\ &= \frac{2r!}{n!}\sum_{g\in G}\binom{\alpha_1+2\alpha_2}{r}.\end{aligned}$$

□

Example 3.5.4. Consider the case where $G = A_6$ acts on $X^{[2]}$. Table 3.17 below gives the number of elements with the same cycle types in G.

Table 3.17: Cycle Types of Elements of A_6

Permutation Type of an Element $g \in G$	Corresponding Cycle Type $(\alpha_1, \alpha_2, \cdots, \alpha_6)$	Corresponding Value of $\alpha_1 + 2\alpha_2$	Corresponding Number of Elements in G
$(a)(b)(c)(d)(e)(f)$	$(6,0,0,0,0,0)$	6	1
$(a)(b)(cd)(ef)$	$(2,2,0,0,0,0)$	6	45
$(a)(b)(c)(def)$	$(3,0,1,0,0,0)$	3	40
$(abc)(def)$	$(0,0,2,0,0,0)$	0	40
$(a)(bcdef)$	$(1,0,0,0,1,0)$	1	144
$(ab)(cdef)$	$(0,1,0,1,0,0)$	2	90
Total			$360 = \|G\|$

The elements with cycle types $(0,0,2,0,0,0)$ and $(1,0,0,0,1,0)$ have no contribution to the number of self-paired suborbits of the action since the expressions $\binom{0}{2}$ and $\binom{1}{2}$ are meaningless. Thus, from Equation 3.5.1 above, the number of self-paired suborbits of G on $X^{[2]}$ is

$$\begin{aligned}\pi &= \frac{2(2!)}{6!}\left[\binom{6}{2}+45\binom{6}{2}+40\binom{3}{2}+90\binom{2}{2}\right] \\ &= \frac{4}{720}[15+675+120+90] \\ &= \frac{900}{180} \\ &= 5.\end{aligned}$$

The 5 suborbits are $\triangle_0$, the trivial suborbit, by default, and $\triangle_1$, $\triangle_2$, $\triangle_5$ and $\triangle_6$ as seen in Example 3.5.1 above.

Example 3.5.5. Similarly, consider the action of $G = A_8$ on $X^{[3]}$.

Table 3.18: Cycle Types of Elements of A_8

Permutation Type of an Element $g \in G$	Corresponding Cycle Type $(\alpha_1, \alpha_2, \cdots, \alpha_8)$	Corresponding Value of $\alpha_1 + 2\alpha_2$	Corresponding Number of Elements in G
$(a)(b)(c)(d)(e)(f)(g)(h)$	$(8,0,0,0,0,0,0,0)$	8	1
$(a)(b)(c)(d)(ef)(gh)$	$(4,2,0,0,0,0,0,0)$	8	210
$(ab)(cd)(ef)(gh)$	$(0,4,0,0,0,0,0,0)$	8	105
$(a)(b)(c)(d)(e)(fgh)$	$(5,0,1,0,0,0,0,0)$	5	112
$(a)(bc)(de)(fgh)$	$(1,2,1,0,0,0,0,0)$	5	1680
$(a)(b)(cde)(fgh)$	$(2,0,2,0,0,0,0,0)$	2	1120
$(a)(b)(cd)(efgh)$	$(2,1,0,1,0,0,0,0)$	4	2520
$(abcd)(efgh)$	$(0,0,0,2,0,0,0,0)$	0	1260
$(a)(b)(c)(defgh)$	$(3,0,0,0,1,0,0,0)$	3	1344
$(abc)(defgh)$	$(0,0,1,0,1,0,0,0)$	0	2688
$(ab)(cdefgh)$	$(0,1,0,0,0,1,0,0)$	2	3360
$(a)(bcdefgh)$	$(1,0,0,0,0,0,1,0)$	1	5760
Total			$20160 = \|G\|$

From the last two columns of Table 3.18 and Equation 3.5.1, the number of self-paired suborbits of G on $X^{[3]}$ is

$$\begin{aligned}
\pi &= \frac{2(3!)}{8!}[\binom{8}{3} + 210\binom{8}{3} + 105\binom{8}{3} \\
&+ 112\binom{5}{3} + 1680\binom{5}{3} + 2520\binom{4}{3} + 1344\binom{3}{3}] \\
&= \frac{1}{3360}[56 + 11760 + 5880 + 1120 + 16800 + 10080 + 1344] \\
&= \frac{1}{3360} \times 47040 \\
&= 14.
\end{aligned}$$

These are the 13 non-trivial self-paired suborbits specified in Example 3.5.2 above, together with the trivial suborbit, for $n = 8$.

CHAPTER FOUR
PROPERTIES AND INVARIANTS OF THE ACTION OF $\mathbf{A_n}$ ON UNORDERED SUBSETS

4.1 Introduction

Primary to the study of any group action are the associated combinatorial properties and invariants. The current chapter provides a thorough examination of these concepts with regard to the action of A_n on $X^{(r)}$, the set of the unordered r-element subsets of $X = \{1, 2, \cdots, n\}$. Transitivity and primitivity of the action are respectively determined in Sections 4.2 and 4.3. Additionally, calculation of the rank and subdegrees of the action is handled in Section 4.4, while examination of pairing of the suborbits of the action is dealt with in Section 4.5.
The action of G on X induces an action of G on $X^{(r)}$. The induced action is defined by $g\{x_1, x_2, \cdots, x_r\} = \{g(x_1), g(x_2), \cdots, g(x_r)\}\ \forall g \in G, \{x_1, x_2, \cdots, x_r\} \in X^{(r)}$. In this case, $|X^{(r)}| = \binom{n}{r} = \frac{n!}{(n-r)!r!}$.

4.2 Transitivity of $\mathbf{A_n}$ on $\mathbf{X^{(2)}}$, $\mathbf{X^{(3)}}$, $\mathbf{X^{(4)}}$ and $\mathbf{X^{(r)}}$

4.2.1 Transitivity of $\mathbf{A_n}$ on $\mathbf{X^{(2)}}$

4.2.1.1 Transitivity of $\mathbf{A_3}$ on $\mathbf{X^{(2)}}$

In this case $G = \{1, (123), (132)\}$ and $X^{(2)} = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$. It is clear that the identity in G fixes the unordered pair $\{1, 2\}$, but the other elements of G move the unordered pair. So, by Definition 1.1.3, $Stab_G\{1, 2\} = \{1\}$. Now, by Theorem 1.1.2,

$$\begin{aligned}
|Orb_G\{1,2\}| &= |G : Stab_G\{1,2\}| \\
&= \frac{|G|}{|Stab_G\{1,2\}|} \\
&= \frac{3}{1} \\
&= 3 \\
&= |X^{(2)}|.
\end{aligned}$$

Hence, by Definition 1.1.4, the action is transitive.

4.2.1.2 Transitivity of $\mathbf{A_4}$ on $\mathbf{X^{(2)}}$

The set under consideration is $X^{(2)} = \{\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\}\}$, and in this case $Stab_G\{1,2\} = \{1,(12)(34)\}$. By Theorem 1.1.2,

$$\begin{aligned} |Orb_G\{1,2\}| &= |G : Stab_G\{1,2\}| \\ &= \frac{|G|}{|Stab_G\{1,2\}|} \\ &= \frac{12}{2} \\ &= 6 \\ &= |X^{(2)}|. \end{aligned}$$

Thus, the action is transitive.

4.2.1.3 Transitivity of $\mathbf{A_5}$ on $\mathbf{X^{(2)}}$

In this case

$$\begin{aligned} X^{(2)} = \; & \{\{1,2\},\{1,3\},\{1,4\},\{1,5\},\{2,3\},\{2,4\}, \\ & \{2,5\},\{3,4\},\{3,5\},\{4,5\}\}. \end{aligned}$$

Now,

$$Stab_G\{1,2\} = \{1,(345),(354),(12)(34),(12)(35),(12)(45)\}$$

and by Theorem 1.1.2,

$$\begin{aligned} |Orb_G\{1,2\}| &= |G : Stab_G\{1,2\}| \\ &= \frac{|G|}{|Stab_G\{1,2\}|} \\ &= \frac{60}{6} \\ &= 10 \\ &= |X^{(2)}|. \end{aligned}$$

So, the action is transitive.

Lemma 4.2.1. The order of the stabilizer in G of an unordered pair $\{1,2\}$ is $\frac{(n-2)!2!}{2}$ for all $n \geq 3$.

Proof. The stabilizer of the unordered pair $\{1,2\}$ is the union of the products of the transposition $(1\ 2)$ by the odd permutations of $\{3,\cdots,n\}$, and the even permutations of $\{3,\cdots,n\}$,

$n \geq 3$. Thus,

$$\begin{aligned} |Stab_G\{1,2\}| &= \frac{(n-2)!}{2} + \frac{(n-2)!}{2} \\ &= \frac{2[(n-2)!]}{2} \\ &= \frac{(n-2)!2!}{2}. \end{aligned}$$

□

Proposition 4.2.1. The group G acts transitively on $X^{(2)}$ for all $n \geq 3$.

Proof. Since $|X^{(2)}| = \binom{n}{2}$, it is sufficient to show that $|Orb_G\{1,2\}| = \binom{n}{2}$. Now, by Theorem 1.1.2 and Lemma 4.2.1,

$$\begin{aligned} |Orb_G\{1,2\}| &= |G : Stab_G\{1,2\}| \\ &= \frac{|G|}{|Stab_G\{1,2\}|} \\ &= \frac{n!/2}{(n-2)!2!/2} \\ &= \frac{n!}{(n-2)!2!} \\ &= \binom{n}{2}. \end{aligned}$$

□

4.2.2 Transitivity of $\mathbf{A_n}$ on $\mathbf{X^{(3)}}$

4.2.2.1 Transitivity of $\mathbf{A_4}$ on $\mathbf{X^{(3)}}$

Now, $X^{(3)} = \{\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\}\}$ and $Stab_G\{1,2,3\} = \{1,(123),(132)\}$. By Theorem 1.1.2,

$$\begin{aligned} |Orb_G\{1,2,3\}| &= |G : Stab_G\{1,2,3\}| \\ &= \frac{|G|}{|Stab_G\{1,2,3\}|} \\ &= \frac{12}{3} \\ &= 4 \\ &= |X^{(3)}|. \end{aligned}$$

Hence, the action is transitive.

4.2.2.2 Transitivity of $\mathrm{A_5}$ on $\mathrm{X^{(3)}}$

The set under consideration is

$$\begin{aligned} X^{(3)} &= \{\{1,2,3\},\{1,2,4\},\{1,2,5\},\{1,3,4\},\{1,3,5\},\{1,4,5\},\\ &\qquad \{2,3,4\},\{2,3,5\},\{2,4,5\},\{3,4,5\}\}. \end{aligned}$$

Also,

$$Stab_G\{1,2,3\} = \{1,(123),(132),(12)(45),(13)(45),(23)(45)\}.$$

By Theorem 1.1.2,

$$\begin{aligned} |Orb_G\{1,2,3\}| &= |G : Stab_G\{1,2,3\}| \\ &= \frac{|G|}{|Stab_G\{1,2,3\}|} \\ &= \frac{60}{6} \\ &= 10 \\ &= |X^{(3)}|. \end{aligned}$$

Thus, the action is transitive.

Lemma 4.2.2. The order of the stabilizer in G of an unordered triple $\{1,2,3\}$ is $\frac{(n-3)!3!}{2}$ for all $n \geq 4$.

Proof. The stabilizer of the unordered triple $\{1,2,3\}$ is the union of the products of the even permutations of $\{1,2,3\}$ by the even permutations of $\{4,\cdots,n\}$, and the products of the odd permutations of $\{1,2,3\}$ by the odd permutations of $\{4,\cdots,n\}$, $n \geq 4$. So,

$$\begin{aligned} |Stab_G\{1,2,3\}| &= \frac{3!}{2}\frac{(n-3)!}{2} + \frac{3!}{2}\frac{(n-3)!}{2} \\ &= \frac{2[(n-3)!3!]}{2.2} \\ &= \frac{(n-3)!3!}{2}. \end{aligned}$$

□

Proposition 4.2.2. The group G acts transitively on $X^{(3)}$ for all $n \geq 4$.

Proof. It suffices to show that $|Orb_G\{1,2,3\}| = |X^{(3)}|$. By Theorem 1.1.2 and Lemma 4.2.2,

$$\begin{aligned}
|Orb_G\{1,2,3\}| &= |G : Stab_G\{1,2,3\}| \\
&= \frac{|G|}{|Stab_G\{1,2,3\}|} \\
&= \frac{n!/2}{(n-3)!3!/2} \\
&= \frac{n!}{(n-3)!3!} \\
&= \binom{n}{3}.
\end{aligned}$$

□

4.2.3 Transitivity of $\mathrm{A_n}$ on $\mathrm{X^{(4)}}$

4.2.3.1 Transitivity of $\mathrm{A_5}$ on $\mathrm{X^{(4)}}$

In this case $X^{(4)} = \{\{1,2,3,4\},\{1,2,3,5\},\{1,2,4,5\},\{1,3,4,5\},\{2,3,4,5\}\}$, and

$$\begin{aligned}
Stab_G\{1,2,3,4\} &= \{1,(123),(132),(134),(143),(124),(142),(234), \\
&\qquad (243),(12)(34),(13)(24),(14)(23)\}.
\end{aligned}$$

By Theorem 1.1.2,

$$\begin{aligned}
|Orb_G\{1,2,3,4\}| &= |G : Stab_G\{1,2,3,4\}| \\
&= \frac{|G|}{|Stab_G\{1,2,3,4\}|} \\
&= \frac{60}{12} \\
&= 5 \\
&= |X^{(4)}|.
\end{aligned}$$

Therefore, the action is transitive, by Definition 1.1.4.

Lemma 4.2.3. The order of the stabilizer in G of an unordered quadruple $\{1,2,3,4\}$ is $\frac{(n-4)!4!}{2}$ for all $n \geq 5$.

Proof. The stabilizer of the unordered quadruple $\{1,2,3,4\}$ is the union of the products of the even permutations of $\{1,2,3,4\}$ by the even permutations of $\{5,\cdots,n\}$, and the products of the odd permutations of $\{1, 2, 3, 4\}$ by the odd permutations of $\{5, \cdots, n\}$, $n \geq 5$. Hence,

$$\begin{aligned} |Stab_G\{1,2,3,4\}| &= \frac{4!}{2}\frac{(n-4)!}{2} + \frac{4!}{2}\frac{(n-4)!}{2} \\ &= \frac{2[(n-4)!4!]}{2.2} \\ &= \frac{(n-4)!4!}{2}. \end{aligned}$$

□

Proposition 4.2.3. The group G acts transitively on $X^{(4)}$ for all $n \geq 5$.

Proof. It is adequate to show that $|Orb_G\{1,2,3,4\}| = \binom{n}{4}$. Now, by Theorem 1.1.2 and Lemma 4.2.3,

$$\begin{aligned} |Orb_G\{1,2,3,4\}| &= |G : Stab_G\{1,2,3,4\}| \\ &= \frac{|G|}{|Stab_G\{1,2,3,4\}|} \\ &= \frac{n!/2}{(n-4)!4!/2} \\ &= \frac{n!}{(n-4)!4!} \\ &= \binom{n}{4}. \end{aligned}$$

□

4.2.4 Transitivity of $\mathbf{A_n}$ on $\mathbf{X^{(r)}}$

Lemma 4.2.4. The order of the stabilizer in G of an unordered r-element subset $\{1, 2, \cdots, r\}$ is $\frac{(n-r)!r!}{2}$ for all $n \geq r+1$.

Proof. The stabilizer of the subset $\{1, 2, \cdots, r\}$ is the union of the products of the even permutations of $\{1, 2, \cdots, r\}$ by the even permutations of $\{r+1, \cdots, n\}$, and the products of the odd permutations of $\{1, 2, \cdots, r\}$ by the odd permutations of $\{r+1, \cdots, n\}$, $n \geq r+1$. Thus,

$$\begin{aligned} |Stab_G\{1,2,\cdots,r\}| &= \frac{r!}{2}\frac{(n-r)!}{2} + \frac{r!}{2}\frac{(n-r)!}{2} \\ &= \frac{2[(n-r)!r!]}{2.2} \\ &= \frac{(n-r)!r!}{2}. \end{aligned}$$

□

Theorem 4.2.1. The group G acts transitively on $X^{(r)}$ for all $n \geq r+1$.

Proof. Since $|X^{(r)}| = \binom{n}{r}$, it is enough to show that $|Orb_G\{1, 2, \cdots, r\}| = \binom{n}{r}$. Now, by Theorem 1.1.2 and Lemma 4.2.4,

$$\begin{aligned}
|Orb_G\{1,2,\cdots,r\}| &= |G : Stab_G\{1,2,\cdots,r\}| \\
&= \frac{|G|}{|Stab_G\{1,2,\cdots,r\}|} \\
&= \frac{n!/2}{(n-r)!r!/2} \\
&= \frac{n!}{(n-r)!r!} \\
&= \binom{n}{r}.
\end{aligned}$$

□

Example 4.2.1. Consider the action of $G = A_{13}$ on $X^{(6)}$. In this case, $|X^{(6)}| = \frac{13!}{7!6!}$. Clearly, $Stab_G\{1, 2, 3, 4, 5, 6\}$ is the union of the products of the even (odd) permutations of the set $\{1, 2, 3, 4, 5, 6\}$ by the even (odd) permutations of the set $\{7, 8, 9, 10, 10, 11, 12, 13\}$. Thus,

$$\begin{aligned}
|Stab_G\{1,2,3,4,5,6\}| &= \frac{6!}{2}\frac{7!}{2} + \frac{6!}{2}\frac{7!}{2} \\
&= 2\left[\frac{6!}{2}\frac{7!}{2}\right] \\
&= \frac{7!6!}{2}.
\end{aligned}$$

Now, by Theorem 1.1.2,

$$\begin{aligned}
|Orb_G\{1,2,3,4,5,6\}| &= |G : Stab_G\{1,2,3,4,5,6\}| \\
&= \frac{|G|}{|Stab_G\{1,2,3,4,5,6\}|} \\
&= \frac{13!/2}{7!6!/2} \\
&= \frac{13!}{7!6!} \\
&= |X^{(6)}|.
\end{aligned}$$

Therefore, by Definition 1.1.4, the action is transitive.

4.3 Primitivity of $\mathbf{A_n}$ on $\mathbf{X^{(2)}}$, $\mathbf{X^{(3)}}$ and $\mathbf{X^{(r)}}$

4.3.1 Primitivity of $\mathbf{A_n}$ on $\mathbf{X^{(2)}}$

From Theorem 4.2.1, the action is transitive for all $n \geq 3$.

4.3.1.1 Primitivity of $\mathbf{A_3}$ on $\mathbf{X^{(2)}}$

Proposition 4.3.1. The action of A_3 on $X^{(2)}$ is primitive.

Proof. Clearly, $|X^{(2)}| = \binom{3}{2} = 3$ is prime. So, the action has only trivial blocks and by Definition 1.1.8, it is primitive. $\square$

4.3.1.2 Primitivity of $\mathbf{A_4}$ on $\mathbf{X^{(2)}}$

Proposition 4.3.2. The action of A_4 on $X^{(2)}$ is imprimitive.

Proof. In this case $X^{(2)} = \{\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\}\}$. Consider the subset $Y = \{\{1,2\},\{3,4\}\}$ of $X^{(2)}$. If $g \in \{1,(12)(34),(13)(24),(14)(23)\}$, then g either fixes the elements of Y or takes each element of Y to the other, so that $gY = Y$. However, if g is any other element of A_4, i.e., $g = (x_1\,x_2\,x_3)$, $x_i \in \{1,2,3,4\}$, $i = 1,2,3$, then it moves each element of Y to an element of $X^{(2)}$ not in Y so that $gY \cap Y = \emptyset$. Hence, Y is a non-trivial block for the action. In general, if $Y = \{\{x_1,x_2\},\{x_3,x_4\}\}$ is a subset of $X^{(2)}$ such that $\{x_1,x_2\} \cap \{x_3,x_4\} = \emptyset$, then Y is a non-trivial block for the action. By Definition 1.1.8, the action is imprimitive. $\square$

4.3.1.3 Primitivity of $\mathbf{A_n}$, $\mathbf{n \geq 5}$ on $\mathbf{X^{(2)}}$

Proposition 4.3.3. The action of G on $X^{(2)}$ is primitive for all $n \geq 5$.

Proof. If $n \geq 5$, then $2 < n-2$. By Theorem 1.1.3, the action is $(n-2)$-transitive and is hence 2-transitive, from Definition 1.1.5. Thus, by Theorem 1.1.6, the action is primitive. $\square$

4.3.2 Primitivity of $\mathbf{A_n}$ on $\mathbf{X^{(3)}}$

From Theorem 4.2.1, the action is transitive for all $n \geq 4$.

4.3.2.1 Primitivity of $\mathbf{A_4}$ on $\mathbf{X^{(3)}}$

Proposition 4.3.4. The action of A_4 on $X^{(3)}$ is primitive.

Proof. The group A_4 acts on $X^{(3)} = \{\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\}\}$. Let $\{x_1,x_2,x_3\}$ and $\{y_1,y_2,y_3\}$ be distinct elements of $X^{(3)}$. Then $|\{x_1,x_2,x_3\} \cap \{y_1,y_2,y_3\}| = 2$, say

$x_1 = y_1 = x^{'}$, $x_2 = y_2 = x^{''}$ and $x_3 \neq y_3$. If $Y = \{\{x^{'}, x^{''}, x_3\}, \{x^{'}, x^{''}, y_3\}\}$, then $g = (x^{'}\, x^{''}\, x_3) \in A_4$ fixes $\{x^{'}, x^{''}, x_3\}$ but moves $\{x^{'}, x^{''}, y_3\}$ to an element not in Y. Hence $gY \cap Y \neq \emptyset$ and $gY \neq Y$. Hence the action lacks a block with two elements. Thus, the action has only trivial blocks and the conclusion is direct. $\square$

4.3.2.2 Primitivity of $\mathbf{A_5}$ on $\mathbf{X^{(3)}}$

Proposition 4.3.5. The action of A_5 on $X^{(3)}$ is primitive.

Proof. In this case,

$$\begin{aligned} X^{(3)} &= \{\{1,2,3\}, \{1,2,4\}, \{1,2,5\}, \{1,3,4\}, \{1,3,5\}, \{1,4,5\}, \\ &\qquad \{2,3,4\}, \{2,3,5\}, \{2,4,5\}, \{3,4,5\}\}. \end{aligned}$$

If Y is a non-trivial block of the action, then $|Y| = 2$ or 5 since $|Y|$ divides $|X^{(3)}|$. Let $\{x_1, x_2, x_3\}$ and $\{y_1, y_2, y_3\}$ be the distinct elements of a subset Y of $X^{(3)}$. If $g = (x_1\, x_2\, x_3)$, then $g\{x_1, x_2, x_3\} \in Y$, but $g\{y_1, y_2, y_3\} \notin Y$ since the elements of Y are not disjoint. So, $gY \cap Y \neq \emptyset$ and $gY \neq Y$. Hence, the action does not have a block consisting of 2 elements only. Now, consider the permutation $g = (1\,2\,3\,4\,5) \in A_5$. In this case $gY_1 = Y_1$ for the subset $Y_1 = \{\{1,2,3\}, \{1,2,5\}, \{1,4,5\}, \{2,3,4\}, \{3,4,5\}\}$, and $gY_2 = Y_2$ for $Y_2 = \{\{1,2,4\}, \{1,3,4\}, \{1,3,5\}, \{2,3,5\}, \{2,4,5\}\}$ with $Y_1 \cap Y_2 = \emptyset$. However, neither of Y_1 or Y_2 is a block since if $g = (1\,2\,3)$, gY_1 and Y_1, also gY_2 and Y_2, overlap partially. Lastly, any other subset Y of $X^{(3)}$ consisting of 5 elements is not a block; this follows from the fact that $gY \cap Y \neq \emptyset$ and $gY \neq Y$ for $g = (1\,2\,3\,4\,5)$. The conclusion is now clear. $\square$

4.3.2.3 Primitivity of $\mathbf{A_6}$ on $\mathbf{X^{(3)}}$

Proposition 4.3.6. The action of A_6 on $X^{(3)}$ is imprimitive.

Proof. It is sufficient to show that $Y = \{\{1,2,3\}, \{4,5,6\}\}$ is a block for the action. If g is a product of an even (odd) permutation of $\{1,2,3\}$ by an even (odd) permutation of $\{4,5,6\}$, then g fixes both elements of Y. On the other hand, if $g = (ax)(bycz)$ where $\{a,b,c\} = \{1,2,3\}$ and $\{x,y,z\} = \{4,5,6\}$, then g takes $\{1,2,3\}$ to $\{4,5,6\}$ and vice versa. In either case, $gY = Y$. Now, let g be any other element of G, i.e., g is a 3-cycle but not an even permutation of $\{1,2,3\}$ or $\{4,5,6\}$, or g is a 5-cycle, or $g = (a\,x)(b\,y)$ where $a, b \in \{1,2,3\}$ and $x, y \in \{4,5,6\}$, or $g = (a\,b)(c\,d)$ where any three of a, b, c and d come from $\{1,2,3\}$ or $\{4,5,6\}$, or g is a product of two 3-cycles neither of which is an even permutation of $\{1,2,3\}$ or $\{4,5,6\}$. Then g moves each element of Y to an element not in Y so that $gY \cap Y = \emptyset$. Hence, Y is a non-trivial block for the action. $\square$

4.3.2.4 Primitivity of $\mathbf{A_n}$, $\mathbf{n \geq 7}$ on $\mathbf{X^{(3)}}$

Proposition 4.3.7. The action of G on $X^{(3)}$ is primitive for all $n \geq 7$.

Proof. Since $2 < n-2$, then by Theorem 1.1.3, the action is $(n-2)$-transitive, and by Definition 1.1.5, it is 2-transitive. The conclusion now follows from Theorem 1.1.6. ☐

4.3.3 Primitivity of $\mathbf{A_n}$ on $\mathbf{X^{(r)}}$

From Theorem 4.2.1, the action is transitive for all $n \geq r+1$.

Theorem 4.3.1. The action of G on $X^{(r)}$ is imprimitive if and only if $n = 2r$.

Proof. It is adequate to prove that G acts on $X^{(r)}$ imprimitively if $n = 2r$ and primitively otherwise. Let $n = 2r$ and let $Y = \{\{x_1, x_2, \cdots, x_r\}, \{y_1, y_2, \cdots, y_r\}\}$ be a subset of $X^{(r)}$ such that $\{x_1, x_2, \cdots, x_r\} \cap \{y_1, y_2, \cdots, y_r\} = \emptyset$. Suppose $g \in Stab_G\{x_1, x_2, \cdots, x_r\}$ or $g \in Stab_G\{y_1, y_2, \cdots, y_r\}$, i.e., g is a product of an even permutation of $\{x_1, x_2, \cdots, x_r\}$ by an even permutation of $\{y_1, y_2, \cdots, y_r\}$ or a product of an odd permutation of $\{x_1, x_2, \cdots, x_r\}$ by an odd permutation of $\{y_1, y_2, \cdots, y_r\}$. Then g fixes each element of Y. On the other hand, let g be a product of an even number of odd cycles of the form $(x_{\alpha_1}\ y_{\beta_1}\ x_{\alpha_2}\ y_{\beta_2} \cdots x_{\alpha_k}\ y_{\beta_k})$, $1 \leq k < r$, $\alpha_i, \beta_j \in \{1, 2, \cdots, r\}$, $i, j = 1, 2, \cdots, k$, where each element of an element of Y belongs to an odd cycle. Then, $g\{x_1, x_2, \cdots, x_r\} = \{y_1, y_2, \cdots, y_r\}$, and vise versa. In either case, $gY = Y$. Now, any other $g \in G$ takes each element of Y to an element of $X^{(r)}$ not in Y so that $gY \cap Y = \emptyset$. Hence, Y is a non-trivial block for the action and, by Definition 1.1.8, the action is imprimitive. Next, suppose $n < 2r$. If n is prime with $n = r+1$, then $r = n-1$ so that $|X^{(r)}| = \begin{pmatrix} n \\ n-1 \end{pmatrix} = n$ and the action will definitely have only trivial blocks. Now, consider the other cases for which $n < 2r$. Clearly, any two elements of $X^{(r)}$ are not disjoint. Hence, if Y is a proper subset of $X^{(r)}$ containing two or more elements, then there exists a permutation $g \in G$ that takes one element of Y to another and the latter to an element not in Y so that $gY \cap Y \neq \emptyset$ and $gY \neq Y$. Thus, the action lacks non-trivial blocks and is therefore primitive. On the other hand, suppose $n > 2r$. Clearly, $2 < n-2$. By Theorem 1.1.3, the action is $(n-2)$-transitive, and by Definition 1.1.5, it is 2-transitive. Thus, by Theorem 1.1.6, the action is primitive. ☐

4.4 Ranks and Subdegrees of $\mathbf{A_n}$ on $\mathbf{X^{(2)}}$, $\mathbf{X^{(3)}}$, $\mathbf{X^{(4)}}$ and $\mathbf{X^{(r)}}$

4.4.1 Rank and Subdegrees of $\mathbf{A_n}$ on $\mathbf{X^{(2)}}$

From Theorem 4.2.1, the action is transitive for all $n \geq 3$. Throughout this subsection, by N it shall mean the set $\{1, 2\}$.

4.4.1.1 Rank and Subdegrees of $\mathbf{A_3}$ on $\mathbf{X^{(2)}}$

In this case $X^{(2)} = \{\{1,2\},\{1,3\},\{2,3\}\}$ and $G_{\{1,2\}} = \{1\}$. Now, $G_{\{1,2\}}$ has orbits whose respective elements contain exactly 2 or 1 element from N:
$\triangle_0 = Orb_{G_{\{1,2\}}}\{1,2\} = \{\{1,2\}\}$, the orbit whose element has both elements from N; the trivial orbit, with $|\triangle_0| = 1 = \binom{2}{2}\binom{1}{0}$.
$\triangle_1 = Orb_{G_{\{1,2\}}}\{1,3\} = \{\{1,3\}\}$ and $\triangle_2 = Orb_{G_{\{1,2\}}}\{2,3\} = \{\{2,3\}\}$, the orbits whose respective elements contain exactly 1 element from N, with $|\triangle_1| = |\triangle_2| = 1$.
Therefore, A_3 acts on $X^{(2)}$ with rank 3 and subdegrees $1,1,1$.

4.4.1.2 Rank and Subdegrees of $\mathbf{A_4}$ on $\mathbf{X^{(2)}}$

In this case
$$X^{(2)} = \{\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\}\}$$
and $G_{\{1,2\}} = \{1,(12)(34)\}$. Now, $G_{\{1,2\}}$ has orbits each of whose every element contains exactly 2, 1, or no element from N:
$\triangle_0 = Orb_{G_{\{1,2\}}}\{1,2\} = \{\{1,2\}\}$, the trivial orbit, where $|\triangle_0| = 1 = \binom{2}{2}\binom{2}{0}$.
$\triangle_1 = Orb_{G_{\{1,2\}}}\{1,3\} = \{\{1,3\},\{2,4\}\}$ and $\triangle_2 = Orb_{G_{\{1,2\}}}\{1,4\} = \{\{1,4\},\{2,3\}\}$, the orbits each of whose every element contains exactly 1 element from N, in which case $|\triangle_1| = |\triangle_2| = 2$.
$\triangle_3 = Orb_{G_{\{1,2\}}}\{3,4\} = \{\{3,4\}\}$, the orbit whose element contains no element from the set N, where $|\triangle_3| = 1 = \binom{2}{0}\binom{2}{2}$.
Therefore, A_4 acts on $X^{(2)}$ with rank 4 and subdegrees $1,1,2,2$.

4.4.1.3 Rank and Subdegrees of $\mathbf{A_5}$ on $\mathbf{X^{(2)}}$

In this case
$$\begin{aligned} X^{(2)} &= \{\{1,2\},\{1,3\},\{1,4\},\{1,5\},\{2,3\},\{2,4\},\\ &\qquad \{2,5\},\{3,4\},\{3,5\},\{4,5\}\} \end{aligned}$$
and $G_{\{1,2\}} = \{1,(12)(34),(12)(35),(12)(45),(345),(354)\}$. Now, $G_{\{1,2\}}$ has orbits each of whose every element contains exactly 2, 1, or no element from N:
$\triangle_0 = Orb_{G_{\{1,2\}}}\{1,2\} = \{\{1,2\}\}$, the trivial orbit, where $|\triangle_0| = 1 = \binom{2}{2}\binom{3}{0}$.
$\triangle_1 = Orb_{G_{\{1,2\}}}\{1,3\} = \{\{1,3\},\{1,4\},\{1,5\},\{2,3\},\{2,4\},\{2,5\}\}$, the orbit whose each element contains exactly 1 element from N, with $|\triangle_1| = 6 = \binom{2}{1}\binom{3}{1}$.

$\triangle_2 = Orb_{G_{\{1,2\}}}\{3,4\} = \{\{3,4\},\{3,5\},\{4,5\}\}$, the orbit whose each element contains no element from N, such that $|\triangle_2| = 3 = \binom{2}{0}\binom{3}{2}$.
Therefore, A_5 acts on $X^{(2)}$ with rank 3 and subdegrees $1, 3, 6$.

4.4.1.4 Rank and Subdegrees of $\mathbf{A_6}$ on $\mathbf{X^{(2)}}$

In this case

$$\begin{aligned} X^{(2)} &= \{\{1,2\},\{1,3\},\{1,4\},\{1,5\},\{1,6\},\{2,3\},\{2,4\},\{2,5\}, \\ &\quad \{2,6\},\{3,4\},\{3,5\},\{3,6\},\{4,5\},\{4,6\},\{5,6\}\} \end{aligned}$$

and

$$\begin{aligned} G_{\{1,2\}} &= \{1,(12)(34),(12)(35),(12)(36),(12)(45),(12)(46),(12)(56), \\ &\quad (34)(56),(35)(46),(36)(45),(345),(354),(346),(364), \\ &\quad (356),(365),(456),(465),(12)(3456),(12)(3465), \\ &\quad (12)(3546),(12)(3564),(12)(3645),(12)(3654)\}. \end{aligned}$$

Now, $G_{\{1,2\}}$ has orbits each of whose every element has exactly 2, 1, or no element from N:
$\triangle_0 = Orb_{G_{\{1,2\}}}\{1,2\} = \{\{1,2\}\}$, the trivial orbit, where $|\triangle_0| = 1 = \binom{2}{2}\binom{4}{0}$.
$\triangle_1 = Orb_{G_{\{1,2\}}}\{1,3\} = \{\{1,3\},\{1,4\},\{1,5\},\{1,6\},\{2,3\},\{2,4\},\{2,5\},\{2,6\}\}$, the orbit whose each element contains exactly 1 element from N, where $|\triangle_1| = 8 = \binom{2}{1}\binom{4}{1}$.
$\triangle_2 = Orb_{G_{\{1,2\}}}\{3,4\} = \{\{3,4\},\{3,5\},\{3,6\},\{4,5\},\{4,6\},\{5,6\}\}$, the orbit whose each element contains no element from N, such that $|\triangle_2| = 6 = \binom{2}{0}\binom{4}{2}$.
Therefore, A_6 acts on $X^{(2)}$ with rank 3 and subdegrees $1, 6, 8$.

4.4.1.5 Rank and Subdegrees of $\mathbf{A_n}$ on $\mathbf{X^{(2)}}$ for $\mathbf{n \geq 5}$

Proposition 4.4.1. The group G acts on $X^{(2)}$ with rank 3 and subdegrees $\binom{2}{2}\binom{n-2}{0}$, $\binom{2}{1}\binom{n-2}{1}$ and $\binom{2}{0}\binom{n-2}{2}$ for all $n \geq 5$.

Proof. Suppose G acts on $X^{(2)}$. Then, $G_{\{1,2\}}$ has orbits each of whose every element contains exactly 2, 1, or no element from N:
$\triangle_0 = Orb_{G_{\{1,2\}}}\{1,2\} = \{\{1,2\}\}$, the trivial orbit. Clearly, $|\triangle_0| = 1 = \binom{2}{2}\binom{n-2}{0}$, the number of ways of selecting 2 objects from a set of 2 distinct objects and 0 objects from a set of $n-2$ distinct objects.

$\triangle_1 = Orb_{G_{\{1,2\}}}\{1,3\} = \{\{1,3\},\{1,4\},\cdots,\{1,n\},\{2,3\},\{2,4\},\cdots,\{2,n\}\}$, the orbit whose each element contains exactly 1 element from N. The length of this orbit is given by $|\triangle_1| = 2(n-2) = \binom{2}{1}\binom{n-2}{1}$, the number of ways of selecting 1 object from a set of 2 distinct objects and 1 object from a set of $n-2$ distinct objects.
$\triangle_2 = Orb_{G_{\{1,2\}}}\{3,4\} = \{\{3,4\},\{3,5\},\cdots,\{3,n\},\{4,5\},\cdots,\{4,n\},\cdots,\{n-1,n\}\}$, the orbit whose each element contains no element from N. The orbit has corresponding length $|\triangle_2| = (n-3)+(n-4)+\cdots+3+2+1 = \frac{(n-2)(n-3)}{2!} = \binom{2}{0}\binom{n-2}{2}$, the number of ways of selecting 0 objects from a set of 2 distinct objects and 2 objects from a set of $n-2$ distinct objects.
Clearly, these orbits are disjoint and summing up the subdegrees,

$$\binom{2}{2}\binom{n-2}{0}+\binom{2}{1}\binom{n-2}{1}+\binom{2}{0}\binom{n-2}{2}=\binom{n}{2}=|X^{(2)}|.$$

Hence each element of $X^{(2)}$ is in some $\triangle_i$ ($i = 0,1,2$) above. So, the rank is 3.
Now, calculations show that the subdegrees are ordered according to increasing magnitude as follows:

$$\begin{cases} \binom{2}{2}\binom{n-2}{0} < \binom{2}{0}\binom{n-2}{2} \leq \binom{2}{1}\binom{n-2}{1} & if\ 5 \leq n \leq 7 \\ \\ \binom{2}{2}\binom{n-2}{0} < \binom{2}{1}\binom{n-2}{1} < \binom{2}{0}\binom{n-2}{2} & if\ n \geq 8. \end{cases}$$

□

4.4.2 Rank and Subdegrees of $\mathbf{A_n}$ on $\mathbf{X^{(3)}}$

From Theorem 4.2.1, the action is transitive for all $n \geq 4$. Throughout this subsection, by N it shall mean the set $\{1,2,3\}$.

4.4.2.1 Rank and Subdegrees of $\mathbf{A_4}$ on $\mathbf{X^{(3)}}$

In this case

$$X^{(3)} = \{\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\}\}$$

and $G_{\{1,2,3\}} = \{1,(123),(132)\}$. Now, $G_{\{1,2,3\}}$ has orbits each of whose every element has exactly 3 or 2 elements from N:
$\triangle_0 = Orb_{G_{\{1,2,3\}}}\{1,2,3\} = \{\{1,2,3\}\}$, the orbit whose element has all the elements from N; the trivial orbit, with $|\triangle_0| = 1 = \binom{3}{3}\binom{1}{0}$.

$\triangle_1 = Orb_{G_{\{1,2,3\}}}\{1,2,4\} = \{\{1,2,4\},\{1,3,4\},\{2,3,4\}\}$, the orbit whose each element has exactly 2 elements from N, where $|\triangle_1| = 3 = \binom{3}{2}\binom{1}{1}$.
Therefore, A_4 acts on $X^{(3)}$ with rank 2 and subdegrees $1, 3$.

4.4.2.2 Rank and Subdegrees of $\mathbf{A_5}$ on $\mathbf{X^{(3)}}$

In this case

$$
\begin{aligned}
X^{(3)} &= \{\{1,2,3\},\{1,2,4\},\{1,2,5\},\{1,3,4\},\{1,3,5\},\{1,4,5\},\\
&\qquad \{2,3,4\},\{2,3,5\},\{2,4,5\},\{3,4,5\}\}
\end{aligned}
$$

and

$$G_{\{1,2,3\}} = \{1,(123),(132),(12)(45),(13)(45),(23)(45)\}.$$

Now, $G_{\{1,2,3\}}$ has orbits each of whose every element has exactly 3, 2, or 1 element from N:
$\triangle_0 = Orb_{G_{\{1,2,3\}}}\{1,2,3\} = \{\{1,2,3\}\}$, the trivial orbit, with $|\triangle_0| = 1 = \binom{3}{3}\binom{2}{0}$.
$\triangle_1 = Orb_{G_{\{1,2,3\}}}\{1,2,4\} = \{\{1,2,4\},\{1,2,5\},\{1,3,4\},\{1,3,5\},\{2,3,4\},\{2,3,5\}\}$, the orbit whose each element contains exactly 2 elements from N, with $|\triangle_1| = 6 = \binom{3}{2}\binom{2}{1}$.
$\triangle_2 = Orb_{G_{\{1,2,3\}}}\{1,4,5\} = \{\{1,4,5\},\{2,4,5\},\{3,4,5\}\}$, the orbit whose each element contains exactly 1 element from N, such that $|\triangle_2| = 3 = \binom{3}{1}\binom{2}{2}$.
Therefore, A_5 acts on $X^{(3)}$ with rank 3 and subdegrees $1, 3, 6$.

4.4.2.3 Rank and Subdegrees of $\mathbf{A_6}$ on $\mathbf{X^{(3)}}$

In this case

$$
\begin{aligned}
X^{(3)} &= \{\{1,2,3\},\{1,2,4\},\{1,2,5\},\{1,2,6\},\{1,3,4\},\{1,3,5\},\{1,3,6\},\{1,4,5\},\\
&\qquad \{1,4,6\},\{1,5,6\},\{2,3,4\},\{2,3,5\},\{2,3,6\},\{2,4,5\},\\
&\qquad \{2,4,6\},\{2,5,6\},\{3,4,5\},\{3,4,6\},\\
&\qquad \{3,5,6\},\{4,5,6\}\}
\end{aligned}
$$

and

$$
\begin{aligned}
G_{\{1,2,3\}} &= \{1,(123),(132),(456),(465),(12)(45),(12)(46),(12)(56),\\
&\qquad (13)(45),(13)(46),(13)(56),(23)(45),(23)(46),(23)(56),\\
&\qquad (123)(456),(123)(465),(132)(456),(132)(465)\}.
\end{aligned}
$$

Now, $G_{\{1,2,3\}}$ has orbits each of whose every element contains exactly 3, 2, 1, or no element from N:
$\triangle_0 = Orb_{G_{\{1,2,3\}}}\{1,2,3\} = \{\{1,2,3\}\}$, the trivial orbit, where $|\triangle_0| = 1 = \binom{3}{3}\binom{3}{0}$.
$\triangle_1 = Orb_{G_{\{1,2,3\}}}\{1,2,4\} = \{\{1,2,4\},\{1,2,5\},\{1,2,6\},\{1,3,4\},\{1,3,5\},$
$\{1,3,6\},\{2,3,4\},\{2,3,5\},\{2,3,6\}\},$
the orbit whose each element contains exactly 2 elements from N, with $|\triangle_1| = 9 = \binom{3}{2}\binom{3}{1}$.
$\triangle_2 = Orb_{G_{\{1,2,3\}}}\{1,4,5\} = \{\{1,4,5\},\{1,4,6\},\{1,5,6\},\{2,4,5\},\{2,4,6\},$
$\{2,5,6\},\{3,4,5\},\{3,4,6\},\{3,5,6\}\},$
the orbit whose each element contains exactly 1 element from N, where $|\triangle_2| = 9 = \binom{3}{1}\binom{3}{2}$.
$\triangle_3 = Orb_{G_{\{1,2,3\}}}\{4,5,6\} = \{\{4,5,6\}\}$, the orbit whose only element contains no element from N, such that $|\triangle_3| = 1 = \binom{3}{0}\binom{3}{3}$.
Therefore, A_6 acts on $X^{(3)}$ with rank 4 and subdegrees $1,1,9,9$.

4.4.2.4 Rank and Subdegrees of $\mathbf{A_7}$ on $\mathbf{X^{(3)}}$

In this case

$$\begin{aligned}
X^{(3)} \quad = \quad & \{\{1,2,3\},\{1,2,4\},\{1,2,5\},\{1,2,6\},\{1,2,7\},\{1,3,4\},\{1,3,5\},\{1,3,6\},\{1,3,7\},\\
& \{1,4,5\},\{1,4,6\},\{1,4,7\},\{1,5,6\},\{1,5,7\},\{1,6,7\},\{2,3,4\},\{2,3,5\},\\
& \{2,3,6\},\{2,3,7\},\{2,4,5\},\{2,4,6\},\{2,4,7\},\{2,5,6\},\{2,5,7\},\\
& \{2,6,7\},\{3,4,5\},\{3,4,6\},\{3,4,7\},\{3,5,6\},\{3,5,7\},\\
& \{3,6,7\},\{4,5,6\},\{4,5,7\},\{4,6,7\},\{5,6,7\}\}.
\end{aligned}$$

and

$$\begin{aligned}G_{\{1,2,3\}} \;=\; & \{1,(123),(132),(456),(457),(465),(467),(475),(476),(567),(576),\\ & (12)(45),(12)(46),(12)(47),(12)(56),(12)(57),(12)(67),(13)(45),\\ & (13)(46),(13)(47),(13)(56),(13)(57),(13)(67),(23)(45),(23)(46),\\ & (23)(47),(23)(56),(23)(57),(23)(67),(45)(67),(46)(57),(47)(56),\\ & (12)(4567),(12)(4576),(12)(4657),(12)(4675),(12)(4756),\\ & (12)(4765),(13)(4567),(13)(4576),(13)(4657),(13)(4675),\\ & (13)(4756),(13)(4765),(23)(4567),(23)(4576),(23)(4657),\\ & (23)(4675),(23)(4756),(23)(4765),(123)(456),(123)(465),\\ & (132)(456),(123)(457),(132)(457),(132)(465),(123)(467),\\ & (132)(467),(123)(475),(132)(475),(123)(476),(132)(476),\\ & (123)(567),(132)(567),(123)(576),(132)(576),\\ & (123)(45)(67),(132)(45)(67),(123)(46)(57),\\ & (132)(46)(57),(123)(47)(56),(132)(47)(56)\}.\end{aligned}$$

Now, $G_{\{1,2,3\}}$ has orbits each of whose every element contains exactly 3, 2, 1, or no element from N:
$\triangle_0 = Orb_{G_{\{1,2,3\}}}\{1,2,3\} = \{\{1,2,3\}\}$, the trivial orbit, with $|\triangle_0| = 1 = \binom{3}{3}\binom{4}{0}$.
$\triangle_1 = Orb_{G_{\{1,2,3\}}}\{1,2,4\} = \{\{1,2,4\},\{1,2,5\},\{1,2,6\},\{1,2,7\},\{1,3,4\},\{1,3,5\},$
$\{1,3,6\},\{1,3,7\},\{2,3,4\},\{2,3,5\},\{2,3,6\},\{2,3,7\}\},$
the orbit whose each element contains exactly 2 elements from N, with $|\triangle_1| = 12 = \binom{3}{2}\binom{4}{1}$.
$\triangle_2 = Orb_{G_{\{1,2,3\}}}\{1,4,5\} = \{\{1,4,5\},\{1,4,6\},\{1,4,7\},\{1,5,6\},\{1,5,7\},\{1,6,7\},$
$\{2,4,5\},\{2,4,6\},\{2,4,7\},\{2,5,6\},\{2,5,7\},\{2,6,7\},$
$\{3,4,5\},\{3,4,6\},\{3,4,7\},\{3,5,6\},\{3,5,7\},\{3,6,7\}\},$
the orbit whose each element contains exactly 1 element from N, and $|\triangle_2| = 18 = \binom{3}{1}\binom{4}{2}$.
$\triangle_3 = Orb_{G_{\{1,2,3\}}}\{4,5,6\} = \{\{4,5,6\},\{4,5,7\},\{4,6,7\},\{5,6,7\}\}$, the orbit whose each element contains no element from N, such that $|\triangle_3| = 4 = \binom{3}{0}\binom{4}{3}$.
Therefore, A_7 acts on $X^{(3)}$ with rank 4 and subdegrees $1, 4, 12, 18$.

4.4.2.5 Rank and Subdegrees of $\mathbf{A_n}$ on $\mathbf{X^{(3)}}$ for $\mathbf{n \geq 6}$

Proposition 4.4.2. The group G acts on $X^{(3)}$ with rank 4 and subdegrees $\binom{3}{3}\binom{n-3}{0}$, $\binom{3}{2}\binom{n-3}{1}$, $\binom{3}{1}\binom{n-3}{2}$ and $\binom{3}{0}\binom{n-3}{3}$ for all $n \geq 6$.

Proof. The group $G_{\{1,2,3\}}$ has orbits each of whose every element contains exactly 3, 2, 1, or no element from N:

$\triangle_0 = Orb_{G_{\{1,2,3\}}}\{1,2,3\} = \{\{1,2,3\}\}$, the trivial orbit. The orbit has corresponding length $|\triangle_0| = 1 = \binom{3}{3}\binom{n-3}{0}$, the number of ways of selecting 3 objects from a set of 3 distinct objects and 0 objects from a set of $n-3$ distinct objects.

$$\begin{aligned}\triangle_1 = Orb_{G_{\{1,2,3\}}}\{1,2,4\} &= \{\{1,2,4\},\{1,2,5\},\cdots,\{1,2,n\},\{1,3,4\},\\ &\{1,3,5\},\ldots,\{1,3,n\},\{2,3,4\},\{2,3,5\},\cdots,\{2,3,n\}\},\end{aligned}$$

the orbit whose each element contains exactly 2 elements from N. The length of the orbit in this case is $|\triangle_1| = 3(n-3) = \binom{3}{2}\binom{n-3}{1}$, the number of ways of selecting 2 objects from a set of 3 distinct objects and 1 object from a set of $n-3$ distinct objects.

$$\begin{aligned}\triangle_2 = Orb_{G_{\{1,2,3\}}}\{1,4,5\} &= \{\{1,4,5\},\ldots,\{1,4,n\},\{1,5,6\},\ldots,\{1,5,n\},\\ &\{1,6,7\},\ldots,\{1,n-1,n\},\{2,4,5\},\ldots,\{2,n-1,n\},\{3,4,5\},\ldots,\{3,n-1,n\}\},\end{aligned}$$

the orbit whose each element contains exactly one element from N. The length of this orbit is $|\triangle_2| = 3[(n-4)+(n-5)+\cdots+2+1] = \frac{3(n-3)(n-4)}{2} = \binom{3}{1}\binom{n-3}{2}$, the number of ways of selecting 1 object from a set of 3 distinct objects and 2 objects from a set containing $n-3$ distinct objects.

$$\begin{aligned}\triangle_3 = Orb_{G_{\{1,2,3\}}}\{4,5,6\} &= \{\{4,5,6\},\cdots,\{4,5,n\},\{4,6,7\},\cdots,\{4,6,n\},\\ &\{4,7,8\},\cdots,\{4,n-1,n\},\{5,6,7\},\cdots,\{5,n-1,n\},\{6,7,8\},\cdots,\{n-2,n-1,n\}\},\end{aligned}$$

the orbit whose each element contains no element from N. It has corresponding length

$$\begin{aligned}|\triangle_3| &= \{[(n-5)+(n-6)+\cdots+2+1]+[(n-6)+(n-7)+\cdots+2+1]\\ &\quad+[(n-7)+(n-8)+\cdots+2+1]+\cdots+[3+2+1]+[2+1]+1\}\\ &= \frac{(n-4)(n-5)}{2}+\frac{(n-5)(n-6)}{2}+\cdots+\frac{3(4)}{2}+\frac{2(3)}{2}+\frac{1(2)}{2}\\ &= \frac{1}{2}\{(n-5)(n-4)+(n-6)(n-5)+\cdots+3(4)+2(3)+1(2)\}\\ &= \frac{1}{2}\left\{\frac{(n-5)(n-4)(n-3)}{3}\right\}\\ &= \frac{(n-3)(n-4)(n-5)}{3!}\\ &= \binom{3}{0}\binom{n-3}{3}.\end{aligned}$$

This is the number of ways of selecting 0 objects from a set of 3 distinct objects and 3 objects from a set of $n-3$ distinct objects.

Clearly, the orbits are distinct and disjoint. Summing up the subdegrees,

$$\sum_{i=0}^{3}|\triangle_i| = \sum_{i=0}^{3}\binom{3}{3-i}\binom{n-3}{i} = \binom{n}{3} = |X^{(3)}|.$$

This shows that each element of $X^{(3)}$ is in exactly one $\triangle_i$ ($i = 0, 1, 2, 3$) above. So, the rank is 4.
Now, calculations show that the subdegrees are ordered according to increasing magnitude as follows:

$$\begin{cases} \binom{3}{3}\binom{n-3}{0} \leq \binom{3}{0}\binom{n-3}{3} < \binom{3}{2}\binom{n-3}{1} \leq \binom{3}{1}\binom{n-3}{2} & if\ 6 \leq n \leq 8 \\ \binom{3}{3}\binom{n-3}{0} < \binom{3}{2}\binom{n-3}{1} < \binom{3}{0}\binom{n-3}{3} \leq \binom{3}{1}\binom{n-3}{2} & if\ 9 \leq n \leq 14 \\ \binom{3}{3}\binom{n-3}{0} < \binom{3}{2}\binom{n-3}{1} < \binom{3}{1}\binom{n-3}{2} < \binom{3}{0}\binom{n-3}{3} & if\ n \geq 15. \end{cases}$$

□

4.4.3 Rank and Subdegrees of $\mathbf{A_n}$ on $\mathbf{X^{(4)}}$

From Theorem 4.2.1, the action is transitive for all $n \geq 5$. Throughout this subsection, by N it shall mean the set $\{1, 2, 3, 4\}$.

4.4.3.1 Rank and Subdegrees of $\mathbf{A_5}$ on $\mathbf{X^{(4)}}$

In this case $X^{(4)} = \{\{1, 2, 3, 4\}, \{1, 2, 3, 5\}, \{1, 2, 4, 5\}, \{1, 3, 4, 5\}, \{2, 3, 4, 5\}\}$ and

$$\begin{aligned} G_{\{1,2,3,4\}} \quad = \quad & \{1, (123), (132), (124), (142), (134), (143), (234), \\ & (243), (12)(34), (13)(24), (14)(23)\}. \end{aligned}$$

Now, $G_{\{1,2,3,4\}}$ has orbits each of whose every element has exactly 4 or 3 elements from N:
$\triangle_0 = Orb_{G_{\{1,2,3,4\}}}\{1, 2, 3, 4\} = \{\{1, 2, 3, 4\}\}$, the orbit whose only element has all elements from N; the trivial orbit, with $|\triangle_0| = 1 = \binom{4}{4}\binom{1}{0}$.
$\triangle_1 = Orb_{G_{\{1,2,3,4\}}}\{1, 2, 3, 5\} = \{\{1, 2, 3, 5\}, \{1, 2, 4, 5\}, \{1, 3, 4, 5\}, \{2, 3, 4, 5\}\}$, the orbit whose each element contains exactly 3 elements from N, with $|\triangle_1| = 4 = \binom{4}{3}\binom{1}{1}$.
Therefore, A_5 acts on $X^{(4)}$ with rank 2 and subdegrees $1, 4$.

4.4.3.2 Rank and Subdegrees of $\mathbf{A_6}$ on $\mathbf{X^{(4)}}$

In this case

$$\begin{aligned} X^{(4)} &= \{\{1,2,3,4\},\{1,2,3,5\},\{1,2,3,6\},\{1,2,4,5\},\{1,2,4,6\},\{1,2,5,6\},\\ &\{1,3,4,5\},\{1,3,4,6\},\{1,3,5,6\},\{1,4,5,6\},\{2,3,4,5\},\\ &\{2,3,4,6\},\{2,3,5,6\},\{2,4,5,6\},\{3,4,5,6\}\} \end{aligned}$$

and

$$\begin{aligned} G_{\{1,2,3,4\}} &= \{1,(123),(132),(124),(142),(134),(143),(234),(243),(12)(34),\\ &(13)(24),(14)(23),(12)(56),(13)(56),(23)(56),(14)(56),\\ &(24)(56),(34)(56),(1234)(56),(1324)(56),\\ &(1342)(56),(1423)(56),(1432)(56)\}. \end{aligned}$$

Now, $G_{\{1,2,3,4\}}$ has orbits each of whose every element contains exactly 4, 3, or 2 elements from N:

$\triangle_0 = Orb_{G_{\{1,2,3,4\}}}\{1,2,3,4\} = \{\{1,2,3,4\}\}$, the trivial orbit, with $|\triangle_0| = 1 = \binom{4}{4}\binom{2}{0}$.

$\triangle_1 = Orb_{G_{\{1,2,3,4\}}}\{1,2,3,5\} = \{\{1,2,3,5\},\{1,2,3,6\},\{1,2,4,5\},\{1,2,4,6\},$
$\{1,3,4,5\},\{1,3,4,6\},\{2,3,4,5\},\{2,3,4,6\}\}$,

the orbit whose each element contains exactly 3 elements from N, and $|\triangle_1| = 8 = \binom{4}{3}\binom{2}{1}$.

$\triangle_2 = Orb_{G_{\{1,2,3,4\}}}\{1,2,5,6\} = \{\{1,2,5,6\},\{1,3,5,6\},\{1,4,5,6\},$
$\{2,3,5,6\},\{2,4,5,6\},\{3,4,5,6\}\}$,

the orbit whose each element contains exactly 2 elements from N, with $|\triangle_2| = 6 = \binom{4}{2}\binom{2}{2}$.

Therefore, A_6 acts on $X^{(4)}$ with rank 3 and subdegrees $1, 6, 8$.

4.4.3.3 Rank and Subdegrees of $\mathbf{A_7}$ on $\mathbf{X^{(4)}}$

In this case

$$\begin{aligned} X^{(4)} &= \{\{1,2,3,4\},\{1,2,3,5\},\{1,2,3,6\},\{1,2,3,7\},\{1,2,4,5\},\{1,2,4,6\},\\ &\{1,2,4,7\},\{1,2,5,6\},\{1,2,5,7\},\{1,2,6,7\},\{1,3,4,5\},\{1,3,4,6\},\\ &\{1,3,4,7\},\{1,3,5,6\},\{1,3,5,7\},\{1,3,6,7\},\{1,4,5,6\},\{1,4,5,7\},\\ &\{1,4,6,7\},\{1,5,6,7\},\{2,3,4,5\},\{2,3,4,6\},\{2,3,4,7\},\{2,3,5,6\},\\ &\{2,3,5,7\},\{2,3,6,7\},\{2,4,5,6\},\{2,4,5,7\},\{2,4,6,7\},\{2,5,6,7\},\\ &\{3,4,5,6\},\{3,4,5,7\},\{3,4,6,7\},\{3,5,6,7\},\{4,5,6,7\}\} \end{aligned}$$

and

$$\begin{aligned} G_{\{1,2,3,4\}} \quad = \quad & \{1, (123), (132), (124), (142), (134), (143), (234), (243), (567), (576), \\ & (12)(34), (13)(24), (14)(23), (12)(56), (12)(57), (12)(67), (13)(56), \\ & (13)(57), (13)(67), (14)(56), (14)(57), (14)(67), (23)(56), (23)(57), \\ & (23)(67), (24)(56), (24)(57), (24)(67), (34)(56), (34)(57), (34)(67), \\ & (123)(567), (123)(576), (132)(567), (132)(576), (124)(567), \\ & (124)(576), (142)(567), (142)(576), (134)(567), (134)(576), \\ & (143)(567), (143)(576), (234)(567), (234)(576), (243)(567), \\ & (243)(576), (12)(34)(567), (12)(34)(576), (13)(24)(567), \\ & (13)(24)(576), (14)(23)(567), (14)(23)(576), (1234)(56), \\ & (1234)(57), (1234)(67), (1243)(56), (1243)(57), \\ & (1243)(67), (1324)(56), (1324)(57), (1324)(67), \\ & (1342)(56), (1342)(57), (1342)(67), (1423)(56), \\ & (1423)(57), (1423)(67), (1432)(56), \\ & (1432)(57), (1432)(67)\}. \end{aligned}$$

Now, $G_{\{1,2,3,4\}}$ has orbits each of whose every element contains exactly 4, 3, 2, or 1 element from N:

$\triangle_0 = Orb_{G_{\{1,2,3,4\}}}\{1,2,3,4\} = \{\{1,2,3,4\}\}$, the trivial orbit, where $|\triangle_0| = 1 = \binom{4}{4}\binom{3}{0}$.

$\triangle_1 = Orb_{G_{\{1,2,3,4\}}}\{1,2,3,5\} = \{\{1,2,3,5\}, \{1,2,3,6\}, \{1,2,3,7\}, \{1,2,4,5\}, \{1,2,4,6\}, \{1,2,4,7\}, \{1,3,4,5\}, \{1,3,4,6\}, \{1,3,4,7\}, \{2,3,4,5\}, \{2,3,4,6\}, \{2,3,4,7\}\}$,

the orbit whose each element contains exactly 3 elements from N, with $|\triangle_1| = 12 = \binom{4}{3}\binom{3}{1}$.

$\triangle_2 = Orb_{G_{\{1,2,3,4\}}}\{1,2,5,6\} = \{\{1,2,5,6\}, \{1,2,5,7\}, \{1,2,6,7\}, \{1,3,5,6\}, \{1,3,5,7\}, \{1,3,6,7\}, \{1,4,5,6\}, \{1,4,5,7\}, \{1,4,6,7\}, \{2,3,5,6\}, \{2,3,5,7\}, \{2,3,6,7\}, \{2,4,5,6\}, \{2,4,5,7\}, \{2,4,6,7\}, \{3,4,5,6\}, \{3,4,5,7\}, \{3,4,6,7\}\}$,

the orbit whose each element contains exactly 2 elements from N, with $|\triangle_2| = 18 = \binom{4}{2}\binom{3}{2}$.

$\triangle_3 = Orb_{G_{\{1,2,3,4\}}}\{1,5,6,7\} = \{\{1,5,6,7\}, \{2,5,6,7\}, \{3,5,6,7\}, \{4,5,6,7\}\}$, the orbit whose each element contains exactly 1 element from N, where $|\triangle_3| = 4 = \binom{4}{1}\binom{3}{3}$.

Therefore, A_7 acts on $X^{(4)}$ with rank 4 and subdegrees $1, 4, 12, 18$.

4.4.3.4 Rank and Subdegrees of $\mathbf{A_8}$ on $\mathbf{X^{(4)}}$

The action of $G_{\{1,2,3,4\}}$ on $X^{(4)}$ has orbits each of whose every element contains exactly 4, 3, 2, 1, or no element from N:

$\triangle_0 = Orb_{G_{\{1,2,3,4\}}}\{1,2,3,4\} = \{\{1,2,3,4\}\}$, the trivial orbit, where $|\triangle_0| = 1 = \binom{4}{4}\binom{4}{0}$.

$$\triangle_1 = Orb_{G_{\{1,2,3,4\}}}\{1,2,3,5\} = \{\{1,2,3,5\},\{1,2,3,6\},\{1,2,3,7\},\{1,2,3,8\}, \{1,2,4,5\},\{1,2,4,6\},\{1,2,4,7\},\{1,2,4,8\},\{1,3,4,5\},\{1,3,4,6\}, \{1,3,4,7\},\{1,3,4,8\},\{2,3,4,5\},\{2,3,4,6\},\{2,3,4,7\},\{2,3,4,8\}\},$$

the orbit whose each element contains exactly 3 elements from N; $|\triangle_1| = 16 = \binom{4}{3}\binom{4}{1}$.

$$\triangle_2 = Orb_{G_{\{1,2,3,4\}}}\{1,2,5,6\} = \{\{1,2,5,6\},\{1,2,5,7\},\{1,2,5,8\},\{1,2,6,7\},\{1,2,6,8\}, \{1,2,7,8\},\{1,3,5,6\},\{1,3,5,7\},\{1,3,5,8\},\{1,3,6,7\},\{1,3,6,8\},\{1,3,7,8\}, \{1,4,5,6\},\{1,4,5,7\},\{1,4,5,8\},\{1,4,6,7\},\{1,4,6,8\},\{1,4,7,8\},\{2,3,5,6\}, \{2,3,5,7\},\{2,3,5,8\},\{2,3,6,7\},\{2,3,6,8\},\{2,3,7,8\},\{2,4,5,6\},\{2,4,5,7\}, \{2,4,5,8\},\{2,4,6,7\},\{2,4,6,8\},\{2,4,7,8\},\{3,4,5,6\},\{3,4,5,7\}, \{3,4,5,8\},\{3,4,6,7\},\{3,4,6,8\},\{3,4,7,8\}\},$$

the orbit whose each element contains exactly 2 elements from N; $|\triangle_2| = 36 = \binom{4}{2}\binom{4}{2}$.

$$\triangle_3 = Orb_{G_{\{1,2,3,4\}}}\{1,5,6,7\} = \{\{1,5,6,7\},\{1,5,6,8\},\{1,5,7,8\},\{1,6,7,8\}, \{2,5,6,7\},\{2,5,6,8\},\{2,5,7,8\},\{2,6,7,8\},\{3,5,6,7\},\{3,5,6,8\}, \{3,5,7,8\},\{3,6,7,8\},\{4,5,6,7\},\{4,5,6,8\},\{4,5,7,8\},\{4,6,7,8\}\},$$

the orbit whose each element contains exactly 1 element from N; $|\triangle_3| = 16 = \binom{4}{1}\binom{4}{3}$.

$\triangle_4 = Orb_{G_{\{1,2,3,4\}}}\{5,6,7,8\} = \{\{5,6,7,8\}\}$, the orbit whose only element contains no element from N, where $|\triangle_4| = 1 = \binom{4}{0}\binom{4}{4}$.

Therefore, A_8 acts on $X^{(4)}$ with rank 5 and subdegrees $1, 1, 16, 16, 36$.

Proposition 4.4.3. The group G acts on $X^{(4)}$ with rank 5 and subdegrees $\binom{4}{4}\binom{n-4}{0}$, $\binom{4}{3}\binom{n-4}{1}$, $\binom{4}{2}\binom{n-4}{2}$, $\binom{4}{1}\binom{n-4}{3}$ and $\binom{4}{0}\binom{n-4}{4}$ for all $n \geq 8$.

Proof. The group $G_{\{1,2,3,4\}}$ has orbits each of whose every element contains exactly 4, 3, 2, 1, or no element from N:

$\triangle_0 = Orb_{G_{\{1,2,3,4\}}}\{1,2,3,4\} = \{\{1,2,3,4\}\}$, the trivial orbit. It has corresponding length $|\triangle_0| = 1 = \binom{4}{4}\binom{n-4}{0}$, the number of ways of selecting 4 objects from a set of 4 objects and 0 objects from a set of $n-4$ distinct objects.

$$\triangle_1 = Orb_{G_{\{1,2,3,4\}}}\{1,2,3,5\} = \{\{1,2,3,5\},\{1,2,3,6\},\cdots,\{1,2,3,n\},\{1,2,4,5\}, \{1,2,4,6\},\cdots,\{1,2,4,n\},\{1,3,4,5\},\cdots,\{1,3,4,n\},\{2,3,4,5\},\cdots,\{2,3,4,n\}\},$$

the orbit whose each element contains exactly 3 elements from N. The length of the orbit in this case is $|\triangle_1| = 4(n-4) = \binom{4}{3}\binom{n-4}{1}$, the number of ways of selecting 3 objects from a set of 4 distinct objects and 1 object from a set of $n-4$ distinct objects.

$\triangle_2 = Orb_{G_{\{1,2,3,4\}}}\{1,2,5,6\} = \{\{1,2,5,6\},\{1,2,5,7\},\cdots,\{1,2,5,n\},$
$\{1,2,6,7\},\cdots,\{1,2,6,n\},\{1,2,7,8\},\cdots,\{1,2,n-2,n-1\},\{1,2,n-2,n\},$
$\{1,2,n-1,n\},\{1,3,5,6\},\cdots,\{1,3,n-1,n\},\{1,4,5,6\},\cdots,\{1,4,n-1,n\},$
$\{2,3,5,6\},\cdots,\{2,3,n-1,n\},\cdots,\{2,4,n-1,n\},\{3,4,5,6\},\cdots,\{3,4,n-1,n\}\}$,

the orbit whose each element contains exactly 2 elements from N. Its corresponding lenthg in this case is

$$\begin{aligned}
|\triangle_2| &= 6[(n-5)+(n-6)+(n-7)+\cdots+3+2+1] \\
&= 6\left\{\frac{(n-5)(n-4)}{2}\right\} \\
&= 3(n-4)(n-5) \\
&= \binom{4}{2}\binom{n-4}{2},
\end{aligned}$$

the number of ways of selecting 2 objects from a set of 4 distinct objects and 2 objects from a set of $n-4$ distinct objects.

$\triangle_3 = Orb_{G_{\{1,2,3,4\}}}\{1,5,6,7\} = \{\{1,5,6,7\},\{1,5,6,8\},\cdots,\{1,5,6,n\},$
$\{1,5,7,8\},\cdots,\{1,5,7,n\},\{1,5,8,9\},\cdots,\{1,5,n-2,n-1\},$
$\{1,5,n-2,n\},\{1,5,n-1,n\},\{1,6,7,8\},\cdots,\{1,n-2,n-1,n\},$
$\{2,5,6,7\},\cdots,\{2,n-2,n-1,n\},\{3,5,6,7\},\cdots,\{3,n-2,n-1,n\},$
$\{4,5,6,7\},\cdots,\{4,n-2,n-1,n\}\}$,

the orbit whose each element contains exactly 1 element from N. It has corresponding length

$$\begin{aligned}
|\triangle_3| &= 4\{[(n-6)+(n-7)+\cdots+3+2+1]+[(n-7)+(n-8)+\cdots+3+2+1] \\
&\quad +[(n-8)+(n-9)+\cdots+3+2+1]+\cdots+[3+2+1]+[2+1]+1\} \\
&= 4\left\{\frac{(n-6)(n-5)}{2}+\frac{(n-7)(n-6)}{2}+\cdots+\frac{3(4)}{2}+\frac{2(3)}{2}+\frac{1(2)}{2}\right\} \\
&= \frac{4}{2}\{(n-6)(n-5)+(n-7)(n-6)+\cdots+3(4)+2(3)+1(2)\} \\
&= \frac{2(n-4)(n-5)(n-6)}{3} \\
&= \binom{4}{1}\binom{n-4}{3},
\end{aligned}$$

the number of ways of selecting 1 object from a set of 4 distinct objects and 3 objects from a set of $n-4$ distinct objects.

$\triangle_4 = Orb_{G_{\{1,2,3,4\}}}\{5,6,7,8\} = \{\{5,6,7,8\},\{5,6,7,9\},\cdots,\{5,6,7,n\},$
$\{5,6,8,9\},\cdots,\{5,6,n-2,n\},\{5,6,n-1,n\},\{5,7,8,9\},\cdots,\{5,7,n-1,n\},$
$\{5,8,9,10\},\cdots,\{5,n-3,n-2,n-1\},\{5,n-3,n-2,n\},\{5,n-2,n-1,n\},$
$\{6,7,8,9\},\cdots,\{6,n-2,n-1,n\},\{7,8,9,10\},\cdots,\{n-3,n-2,n-1,n\}\}$,

the orbit whose each element contains no element from N. The length of this orbit is given by

$$
\begin{aligned}
|\triangle_4| &= \{\{[(n-7)+(n-8)+\cdots+2+1]+[(n-8)+(n-9)+\cdots+2+1]\\
&\quad +[(n-9)+(n-10)+\cdots+2+1]+\cdots+[3+2+1]+[2+1]+1\}\\
&\quad +\{[(n-8)+(n-9)+\cdots+2+1]+[(n-9)+(n-10)+\cdots+1]\\
&\quad +[(n-10)+(n-11)+\cdots+2+1]+\cdots+[3+2+1]+[2+1]+1\}\\
&\quad +\{[(n-9)+(n-10)+\cdots+2+1]+[(n-10)+(n-11)+\cdots+1]\\
&\quad +[(n-11)+(n-12)+\cdots+2+1]+\cdots+[3+2+1]+[2+1]+1\}\\
&\quad +\cdots+\{[3+2+1]+[2+1]+1\}+\{[2+1]+1\}+1\}\\
&= [\frac{(n-7)(n-6)}{2}+\frac{(n-8)(n-7)}{2}+\cdots+\frac{1(2)}{2}]+[\frac{(n-8)(n-7)}{2}\\
&\quad +\frac{(n-9)(n-8)}{2}+\cdots+\frac{1(2)}{2}]+[\frac{(n-9)(n-8)}{2}+\frac{(n-10)(n-9)}{2}\\
&\quad +\cdots+\frac{1(2)}{2}]+[\frac{(n-10)(n-9)}{2}+\frac{(n-11)(n-10)}{2}+\cdots+\frac{1(2)}{2}]\\
&\quad +\cdots+[\frac{3(4)}{2}+\frac{2(3)}{2}+\frac{1(2)}{2}]+[\frac{2(3)}{2}+\frac{1(2)}{2}]+[\frac{1(2)}{2}]\\
&= \frac{1}{2}[(n-7)(n-6)+(n-8)(n-7)+\cdots+1(2)]+\frac{1}{2}[(n-8)(n-7)\\
&\quad +(n-9)(n-8)+\cdots+1(2)]+\frac{1}{2}[(n-9)(n-8)+(n-10)(n-9)\\
&\quad +\cdots+1(2)]+\frac{1}{2}[3(4)+2(3)+1(2)]+\frac{1}{2}[2(3)+1(2)]+\frac{1}{2}[1(2)]\\
&= \frac{1}{2}\left\{\frac{(n-7)(n-6)(n-5)}{3}+\frac{(n-8)(n-7)(n-6)}{3}+\cdots+\frac{1(2)(3)}{3}\right\}\\
&= \frac{1}{6}[(n-7)(n-6)(n-5)+(n-8)(n-7)(n-6)+\cdots+1(2)(3)]\\
&= \frac{(n-7)(n-6)(n-5)(n-4)}{6\times 4}\\
&= \frac{(n-4)(n-5)(n-6)(n-7)}{4!}\\
&= \binom{4}{0}\binom{n-4}{4}.
\end{aligned}
$$

This is the number of ways of selecting 0 objects from a set of 4 distinct objects and 4 objects from a set of $n-4$ distinct objects.

Clearly, the 5 suborbits are distinct and disjoint. Summing up their subdegrees,

$$\sum_{i=0}^{4}|\triangle_i| = \sum_{i=0}^{4}\binom{4}{4-i}\binom{n-4}{i} = \binom{n}{4} = |X^{(4)}|.$$

This shows that each element of $X^{(4)}$ is in exactly one $\triangle_i$ ($i=0,1,2,3,4$) above. So, the rank of the action is 5.

Now, calculations show that the subdegrees are ordered according to increasing magnitude as follows:

$$\begin{cases} \binom{4}{4}\binom{n-4}{0} \leq \binom{4}{0}\binom{n-4}{4} < \binom{4}{3}\binom{n-4}{1} & \\ < \binom{4}{1}\binom{n-4}{3} < \binom{4}{2}\binom{n-4}{2} & if\ 8 \leq n \leq 10 \\ & \\ \binom{4}{4}\binom{n-4}{0} < \binom{4}{3}\binom{n-4}{1} < \binom{4}{0}\binom{n-4}{4} & \\ \leq \binom{4}{2}\binom{n-4}{2} < \binom{4}{1}\binom{n-4}{3} & if\ 11 \leq n \leq 15 \\ & \\ \binom{4}{4}\binom{n-4}{0} < \binom{4}{3}\binom{n-4}{1} < \binom{4}{2}\binom{n-4}{2} & \\ < \binom{4}{0}\binom{n-4}{4} \leq \binom{4}{1}\binom{n-4}{3} & if\ 16 \leq n \leq 23 \\ & \\ \binom{4}{4}\binom{n-4}{0} < \binom{4}{3}\binom{n-4}{1} < \binom{4}{2}\binom{n-4}{2} & \\ < \binom{4}{1}\binom{n-4}{3} < \binom{4}{0}\binom{n-4}{4} & if\ n \geq 24. \end{cases}$$

□

4.4.4 Rank and Subdegrees of $\mathbf{A_n}$ on $\mathbf{X^{(r)}}$

The results stated and proved in this subsection, on the rank and subdegrees of the action of G on $X^{(r)}$, are derived from the observations made in Subsections 4.4.1 through 4.4.3, above.

Lemma 4.4.1. If the action of G on $X^{(r)}$ has a suborbit whose each element has exactly i ($i = 0, 1, 2, \cdots, r$) elements from the set $N = \{1, 2, \cdots, r\}$, then $n \geq 2r - i$, in which case the rank of the action is at least $r - i + 1$.

Proof. Let $\triangle_{r-i}$ be the orbit whose each element contains exactly i elements from N. Then once the first i elements of an element of $\triangle_{r-i}$ have been selected from N, there remain $r - i$ elements to be selected from the remaining $n - r$ elements of X. For this to happen, it is required that $r - i \leq n - r$, which becomes $n \geq 2r - i$ on rewriting. Accordingly, $G_{\{1,2,\cdots,r\}}$ has orbits each of whose every element has exactly $r,\ r-1,\ r-2, \cdots,\ i+2,\ i+1,$ or i elements from N. These are
$\triangle_0 = Orb_{G_{\{1,2,\cdots,r\}}}\{1, 2, \cdots, r\}$, the orbit whose only element contains exactly r elements from N (the trivial orbit), where $|\triangle_0| = \binom{r}{r}\binom{n-r}{0}$, the number of ways of selecting r objects from r distinct objects and no object from $n - r$ distinct objects,
$\triangle_1 = Orb_{G_{\{1,2,\cdots,r\}}}\{1, 2, \cdots, r-1, r+1\}$, the orbit whose each element contains exactly

$r-1$ elements from N, with $|\triangle_1| = \binom{r}{r-1}\binom{n-r}{1}$, the number of ways of selecting $r-1$ objects from r distinct objects and 1 object from $n-r$ distinct objects, and

$\triangle_2 = Orb_{G_{\{1,2,\cdots,r\}}}\{1,2,\cdots,r-2,r+1,r+2\}$, the orbit whose each element contains exactly $r-2$ elements from N, such that $|\triangle_2| = \binom{r}{r-2}\binom{n-r}{2}$, the number of ways of selecting $r-2$ objects from r distinct objects and 2 objects from $n-r$ distinct objects.

The intermediate orbits $\triangle_3, \cdots, \triangle_{r-i-2}$ are described in an analogous manner. The remaining orbits are

$\triangle_{r-i-1} = Orb_{G_{\{1,2,\cdots,r\}}}\{1,2,\cdots,i,i+1,r+1,r+2,\cdots,2r-i-1\}$, the orbit whose each element contains exactly $i+1$ elements from N; the length corresponding to this orbit is $|\triangle_{r-i-1}| = \binom{r}{i+1}\binom{n-r}{r-i-1}$, the number of ways of selecting $i+1$ objects from r distinct objects and $r-i-1$ objects from $n-r$ distinct objects, and

$\triangle_{r-i} = Orb_{G_{\{1,2,\cdots,r\}}}\{1,2,\cdots,i,r+1,r+2,\cdots,2r-i\}$, the orbit whose each element contains exactly i elements from N, where $|\triangle_{r-i}| = \binom{r}{i}\binom{n-r}{r-i}$, the number of ways of selecting i objects from r distinct objects and $r-i$ objects from $n-r$ distinct objects.

Clearly, the orbits do not overlap partially and are $r-i+1$ in number. $\square$

Theorem 4.4.1. The rank of G on $X^{(r)}$ is $r+1$ if and only if $n \geq 2r$.

Proof. Suppose $n \geq 2r$. This corresponds to $i=0$ in Lemma 4.4.1 and it then follows from the lemma that the stabilizer $G_{\{1,2,\cdots,r\}}$ has orbits each of whose every element contains exactly $r,\ r-1,\ r-2,\cdots,\ 2,\ 1$, or no element from $N=\{1,2,\cdots,r\}$. The $r+1$ suborbits of G are $\triangle_0,\triangle_1,\triangle_2,\cdots,\triangle_{r-2},\triangle_{r-1}$, and $\triangle_r$ respectively. Now, to prove that G has exactly $r+1$ suborbits, it suffices to show that $\{\triangle_0,\triangle_1,\triangle_2,\cdots,\triangle_{r-1},\triangle_r\}$ is a partition of $X^{(r)}$. Clearly, $\triangle_{r-i} \neq \emptyset$ for each $i = 0,1,2,\cdots,r$ and $\triangle_i \cap \triangle_j = \emptyset$ unless $i=j$ $(i,j = 0,1,2,\cdots,r)$. Also,

$$\sum_{i=0}^{r} |\triangle_i| = \sum_{i=0}^{r} \binom{r}{r-i}\binom{n-r}{i} = \binom{n}{r} = |X^{(r)}|$$

so that $\bigcup_{i=0}^{r} \triangle_{r-i} = X^{(r)}$. Thus $\{\triangle_0,\triangle_1,\triangle_2,\cdots,\triangle_{r-1},\triangle_r\}$ partitions $X^{(r)}$.

Conversely, suppose the rank is $r+1$. Then there exists a suborbit $\triangle_r$ corresponding to $i=0$ in Lemma 4.4.1. The length of this suborbit is $\binom{r}{0}\binom{n-r}{r}$ wherein the factor $\binom{n-r}{r}$ is defined only if $n-r \geq r$, which becomes $n \geq 2r$ on rewriting. $\square$

Theorem 4.4.2. If $n \geq 2r$, the length of the suborbit $\triangle_i\,(i = 0,1,2,\cdots,r)$ whose each element has exactly $r-i$ elements from $\{1,2,\cdots,r\}$ is $\binom{r}{r-i}\binom{n-r}{i}$. Further more, $|\triangle_i| < |\triangle_{i+1}|$, i.e. $\binom{r}{r-i}\binom{n-r}{i} < \binom{r}{r-i-1}\binom{n-r}{i+1}$, for all $n \geq r(r+2)$.

Proof. From the proof of Theorem 4.4.1, $|\triangle_i| = \binom{r}{r-i}\binom{n-r}{i}$, the number of ways of selecting $r-i$ objects from r distinct objects and i objects from $n-r$ distinct objects.
The proof of the other part of the theorem is by mathematical induction. If $r=2$, then from Proposition 4.4.1, $\binom{2}{2-i}\binom{n-2}{i} < \binom{2}{2-i-1}\binom{n-2}{i+1}$, $(i=0,1,2)$, for all $n \geq 8$. So, the statement is true for $r=2$. Now, suppose the statement is true if $r=k$ for an integer $k \geq 3$. That is, if $n \geq k(k+2)$,

$$\binom{k}{k-i}\binom{n-k}{i} < \binom{k}{k-i-1}\binom{n-k}{i+1}$$

$$\Rightarrow \frac{k!}{i!(k-i)!}\frac{(n-k)!}{(n-k-i)!i!} < \frac{k!}{(i+1)!(k-i-1)!}\frac{(n-k)!}{(n-k-i-1)!(i+1)!}$$

$$\Rightarrow \quad \frac{k!}{i!(k-i)!}\frac{(n-k)(n-k-1)!}{(n-k-i)(n-k-i-1)!i!} < \frac{k!}{(i+1)!(k-i-1)!}\frac{(n-k)(n-k-1)!}{(n-k-i-1)(n-k-i-2)!(i+1)!}$$

$$\Rightarrow \quad \frac{k!}{i!(k-i)!}\frac{(n-k-1)!}{(n-k-i-1)!i!}\left(\frac{n-k}{n-k-i}\right) < \frac{k!}{(i+1)!(k-i-1)!}\frac{(n-k-1)!}{(n-k-i-2)!(i+1)!}\left(\frac{n-k}{n-k-i-1}\right)$$

$$\Rightarrow \quad \frac{k!}{i!(k-i)!}\frac{(n-k-1)!}{(n-k-i-1)!i!} < \frac{k!}{(i+1)!(k-i-1)!}\frac{(n-k-1)!}{(n-k-i-2)!(i+1)!}\left(\frac{n-k-i}{n-k-i-1}\right).$$

For $r=k+1$, the aim is to show that

$$\binom{k+1}{k-i+1}\binom{n-k-1}{i} < \binom{k+1}{k-i}\binom{n-k-1}{i+1}$$

$$\Rightarrow \frac{(k+1)!}{i!(k-i+1)!}\frac{(n-k-1)!}{(n-k-i-1)!i!} < \frac{(k+1)!}{(i+1)!(k-i)!}\frac{(n-k-1)!}{(n-k-i-2)!(i+1)!}$$

$$\Rightarrow \quad \frac{(k+1)k!}{i!(k-i+1)(k-i)!}\frac{(n-k-1)!}{(n-k-i-1)!i!} < \frac{(k+1)k!}{(i+1)!(k-i)(k-i-1)!}\frac{(n-k-1)!}{(n-k-i-2)!(i+1)!}$$

$$\Rightarrow \quad \frac{k!}{i!(k-i)!}\frac{(n-k-1)!}{(n-k-i-1)!i!}\left(\frac{k+1}{k-i+1}\right) <$$
$$\frac{k!}{(i+1)!(k-i-1)!}\frac{(n-k-1)!}{(n-k-i-2)!(i+1)!}\left(\frac{k+1}{k-i}\right)$$

$$\Rightarrow \quad \frac{k!}{i!(k-i)!}\frac{(n-k-1)!}{(n-k-i-1)!i!} <$$
$$\frac{k!}{(i+1)!(k-i-1)!}\frac{(n-k-1)!}{(n-k-i-2)!(i+1)!}\left(\frac{k-i+1}{k-i}\right).$$

Clearly, $\frac{n-k-i}{n-k-i-1} > 1$ and $\frac{k-i+1}{k-i} > 1$. Now, since $n \geq k(k+2)$ which on rewriting gives $n-2k \geq k^2$, then

$$\begin{aligned}(n-k-i)-(k-i+1) &= (n-2k)-1 \\ &\geq k^2-1 \\ &> 0\end{aligned}$$

since $k \geq 3$. So, $n-k-i > k-i+1$ and hence $\frac{k-i+1}{k-i} > \frac{n-k-i}{n-k-i-1}$. From the inductive hypothesis, that is,

$$\frac{k!}{i!(k-i)!}\frac{(n-k-1)!}{(n-k-i-1)!i!} <$$
$$\frac{k!}{(i+1)!(k-i-1)!}\frac{(n-k-1)!}{(n-k-i-2)!(i+1)!}\left(\frac{n-k-i}{n-k-i-1}\right),$$

and the fact that $\frac{k-i+1}{k-i} > \frac{n-k-i}{n-k-i-1}$, then

$$\frac{k!}{i!(k-i)!}\frac{(n-k-1)!}{(n-k-i-1)!i!} <$$
$$\frac{k!}{(i+1)!(k-i-1)!}\frac{(n-k-1)!}{(n-k-i-2)!(i+1)!}\left(\frac{k-i+1}{k-i}\right).$$

This proves that

$$\binom{k+1}{k-i+1}\binom{n-k-1}{i} < \binom{k+1}{k-i}\binom{n-k-1}{i+1}.$$

So, the statement is true for $r = k+1$ whenever true for $r = k$. Therefore, by the principle of mathematical induction, the statement is true for all $r \geq 2$. □

Remark 4.4.1. Theorem 4.4.1 and part of Theorem 4.4.2 hold only for $n \geq 5$ when $r = 2$. This is clear from Proposition 4.4.1.

4.5 Pairing of Suborbits of $\mathbf{A_n}$ on $\mathbf{X^{(r)}}$

Since $|G|$ is even, then by Theorem 1.1.4, the action has non-trivial suborbits that are self-paired.

Theorem 4.5.1. The suborbits of the action of G on $X^{(r)}$, except for some non-trivial suborbits associated with the actions of A_3 and A_4 on $X^{(2)}$, are self-paired.

Proof. Consider the general suborbit $\triangle_{r-i}$ $(i = 0, 1, \cdots, r)$ whose each element contains exactly i elements from the set $\{1, 2, \cdots, r\}$. Then

$$\{1, 2, \cdots, i, r+1, r+2, \cdots, 2r-i-1, 2r-i\} \in \triangle_{r-i}.$$

If r and i are both even or both odd, then

$$g = (i+1 \;\; r+1)(i+2 \;\; r+2)\cdots(r-1 \;\; 2r-i-1)(r \;\; 2r-i) \in G$$

and if one of r and i is even, and the other odd, then

$$g = (1 \;\; 2)(i+1 \;\; r+1)(i+2 \;\; r+2)\cdots(r-1 \;\; 2r-i-1)(r \;\; 2r-i) \in G.$$

In any of these cases

$$\begin{aligned} & g\{1, 2, \cdots, i, r+1, r+2, \cdots, 2r-i-1, 2r-i\} \\ = & \\ & \{1, 2, \cdots, i, i+1, i+2, \cdots, r-1, r\} \in g\triangle_{r-i} \end{aligned}$$

and

$$\begin{aligned} & g\{1, 2, \cdots, i, i+1, i+2, \cdots, r-1, r\} \\ = & \\ & \{1, 2, \cdots, i, r+1, r+2, \cdots, 2r-i-1, 2r-i\} \in \triangle_{r-i} \end{aligned}$$

so that by Definition 1.1.7, $\triangle^*_{r-i} = \triangle_{r-i}$ (it is easy to verify that no such g exists for cases where $n = 3$ and $n = 4$ while $r = 2$, $i = 1$; see Remark 4.5.1 and Example 4.5.1 below for more details). □

Remark 4.5.1. Theorem 4.5.1, partially fails for the action of A_3 on $X^{(2)}$. The two non-trivial suborbits of the action, found in Subsubsection 4.4.1.1, are paired (see Example 4.5.1 below). The theorem also partially fails for the action of A_4 on $X^{(2)}$ where the two suborbits whose each element contains exactly one of 1 and 2, found in Subsubsection 4.4.1.2, are paired (again see Example 4.5.1 for further details).

Example 4.5.1. The non-trivial suborbits of the action of $G = A_3$ on $X^{(2)}$ are $\triangle_1 = Orb_{G_{\{1,2\}}}\{1,3\} = \{\{1,3\}\}$ and $\triangle_2 = Orb_{G_{\{1,2\}}}\{2,3\} = \{\{2,3\}\}$. Now, consider $g = (1\ 2\ 3) \in G$, then $g\{1,3\} = \{1,2\} \in g\triangle_1$ but $g\{1,2\} = \{2,3\} \in \triangle_2$. So, $\triangle_1^* = \triangle_2$. Similarly, if A_4 acts on $X^{(2)}$, the corresponding non-trivial suborbits $\triangle_1$, $\triangle_2$ and $\triangle_3$ seen in Subsubsection 4.4.1.2, are such that $\triangle_1^* = \triangle_2$ but $\triangle_3^* = \triangle_3$.

Example 4.5.2. The action of A_5 on $X^{(2)}$ has two non-trivial suborbits $\triangle_1$ and $\triangle_2$ (see Subsubsection 4.4.1.3). Now, considering $g_1 = (23)(45)$, then $g_1\{1,3\} = \{1,2\} \in g_1\triangle_1$ and $g_1\{1,2\} = \{1,3\} \in \triangle_1$. So, $\triangle_1$ is self-paired. Also, $g_2 = (13)(24)$ takes $\{3,4\} \in \triangle_2$ to $\{1,2\}$ and vice versa so that $\triangle_2$ is self-paired. Similarly, by Definition 1.1.7, the three non-trivial suborbits $\triangle_1$, $\triangle_2$ and $\triangle_3$ of the action of A_6 on $X^{(3)}$ (see Subsubsection 4.4.2.3) are self-paired. This is clear because $g_3 = (12)(34)$ takes $\{1,2,4\} \in \triangle_1$ to $\{1,2,3\}$ and vice versa, $g_4 = (24)(35)$ takes $\{1,4,5\} \in \triangle_2$ to $\{1,2,3\}$ and vice versa, while $g_5 = (1425)(36)$ takes $\{4,5,6\} \in \triangle_3$ to $\{1,2,3\}$ and vice versa.

CHAPTER FIVE
SUBORBITAL GRAPHS OF THE ACTION OF $\mathbf{A_n}$ ON ORDERED SUBSETS

5.1 Introduction

The properties of the combinatorial structures associated with a group action form an essential component of the study of the group action. The aim of this chapter is to construct the suborbital graphs associated with the action of A_n on $X^{[r]}$ and also give an in-depth analysis of these graphs with regard to concepts such as directedness, connectedness, number of components in a disconnected graph, vertex degree, and girth. Construction and analysis of suborbital graphs corresponding to the six non-trivial suborbits of the action of A_n on $X^{[2]}$ for $n \geq 6$, seen in Subsection 3.4.1, is done in Section 5.2. On the other hand, the construction and analysis of the graphs corresponding to more general action of A_n on $X^{[r]}$ takes place in Section 5.3.

5.2 Suborbital Graphs of $\mathbf{A_n}$ on $\mathbf{X^{[2]}}$

Consider the action of $A_n\,(n \geq 6)$ on $X^{[2]}$ (see the action in Subsubsection 3.4.1.4). Then the suborbits $\triangle_1, \triangle_2, \triangle_5$ and $\triangle_6$ are self-paired while $\triangle_3$ and $\triangle_4$ are paired (see this pairing in Example 3.5.1). Thus, by Theorem 1.1.11, the suborbital graphs corresponding to $\triangle_1, \triangle_2, \triangle_5$ and $\triangle_6$ are undirected while those corresponding to $\triangle_3$ and $\triangle_4$ are directed.
A suborbital graph corresponding to a suborbit of the action has $X^{[2]}$ as its vertex set. Now, the six non-trivial graphs of the action are constructed and described as follows:
(i) The suborbital O_1 corresponding to the suborbit $\triangle_1$ is

$$O_1 = \{(g[1,2], g[2,1]) | g \in G, [2,1] \in \triangle_1\}.$$

So, the corresponding suborbital graph Γ_1 has an edge from vertex $[u,v]$ to vertex $[x,y]$ if and only if $u = y$ and $v = x$. The graph is undirected since if $[x,y]$ is in O_1, so does $[y,x]$. It is disconnected since there is no path from $[1,2]$ to $[1,3]$; a component of the graph is a tree with two leaves. Thus, it is a regular forest of degree 1 consisting of $\frac{|X^{[2]}|}{2} = \frac{1}{2}\left({}_nP_2\right) = \binom{n}{2}$ trees. Its girth is zero, by the definition of a tree. The graph is bipartite with one part corresponding to all ordered pairs $[u,v]$ such that $u < v$, and the other $[x,y]$ such that $x > y$.

Example 5.2.1. For $n = 6$, the suborbital graph Γ_1 corresponding to the suborbit $\triangle_1$ is given as in Figure 5.1 below.

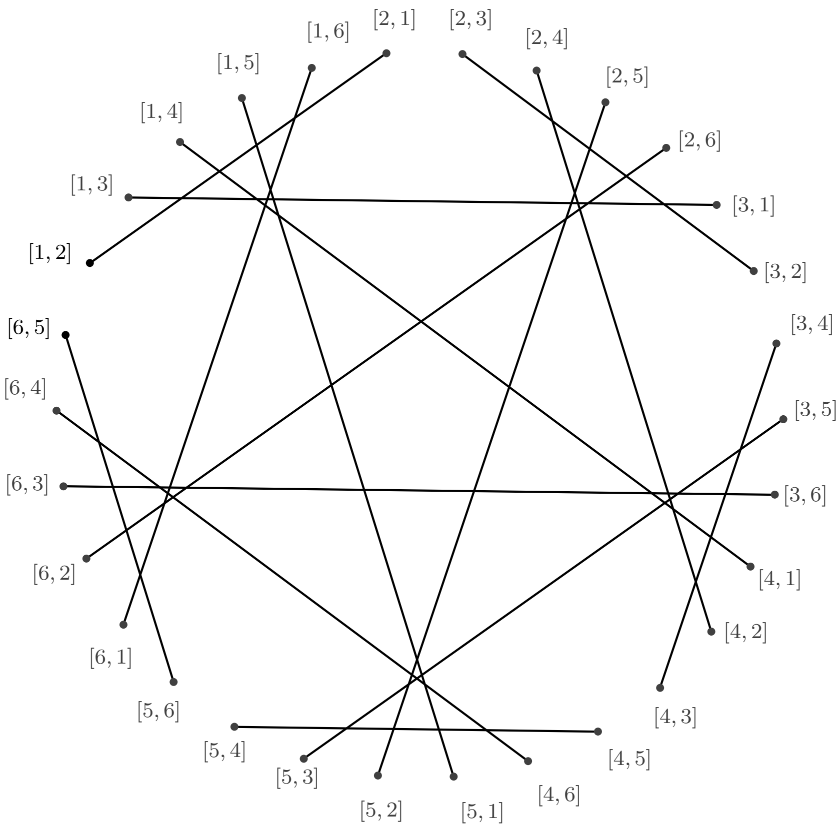

Figure 5.1: Suborbital Graph Γ_1 of A_6 on $X^{[2]}$

(ii) The suborbital O_2 corresponding to the suborbit $\triangle_2$ is

$$O_2 = \{(g[1,2], g[1,3]) | g \in G, [1,3] \in \triangle_2\}.$$

So, the corresponding suborbital graph Γ_2 has an edge from $[u, v]$ to $[x, y]$ if and only if $u = x$ but $v \neq y$. Now, if $([x, y], [x, z]) \in O_2$, then $([x, z], [x, y]) \in O_2$ also. So, Γ_2 is undirected and hence self-paired. To show that it is regular of degree $n - 2$, without loss of generality, the vertex $[1, 2]$ is adjacent to each of the $n-2$ vertices $[1, 3], [1, 4], \cdots, [1, n-1]$, and $[1, n]$. It is disconnected since there does not exist a path between the vertices $[1, 2]$ and $[2, 1]$. A connected component consists of the $n - 1$ vertices of the form $[x, y]$ for a fixed $x \in X$ and $y \in X - \{x\}$. Thus, the number of connected components is $\frac{|X^{[2]}|}{n-1} = \frac{1}{n-1}\left({}_nP_2\right) = \frac{2}{n-1}\binom{n}{2}$. Moreover, it has girth 3 since the vertices $[1, 2]$, $[1, 3]$ and $[1, 4]$ are pairwise adjacent.

Example 5.2.2. If $n = 6$, the suborbital graph Γ_2 corresponding to the suborbit $\triangle_2$ is given as in Figure 5.2 below.

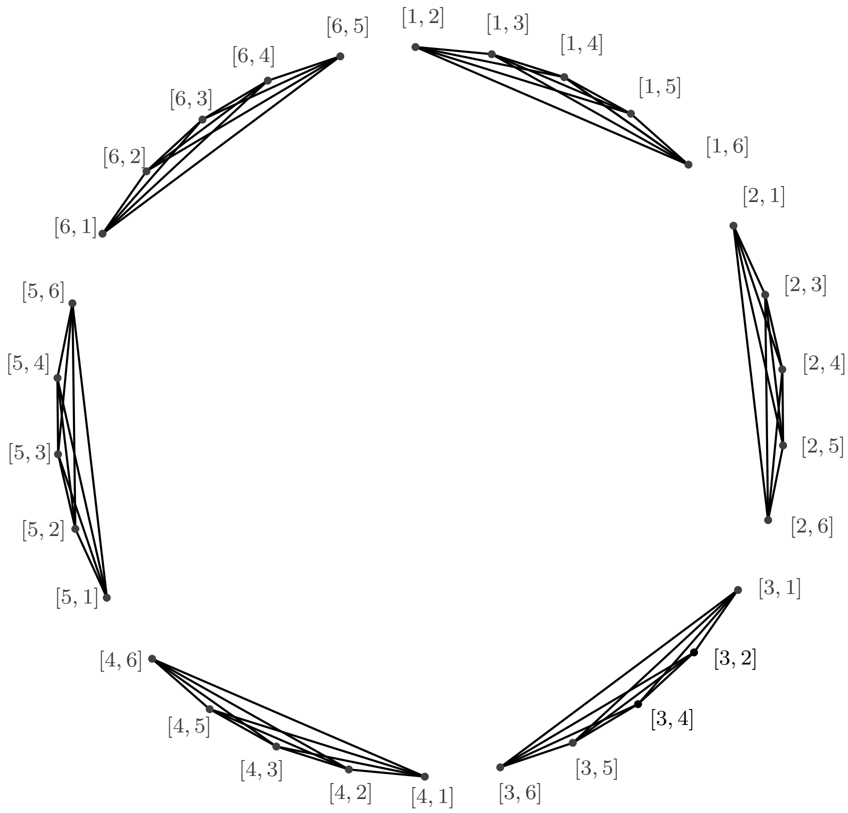

Figure 5.2: Suborbital Graph Γ_2 of A_6 on $X^{[2]}$

(iii) The suborbital O_3 corresponding to the suborbit $\triangle_3$ is

$$O_3 = \{(g[1,2], g[3,1])|g \in G, [3,1] \in \triangle_3\}.$$

So, the corresponding suborbital graph Γ_3 has an edge from $[u,v]$ to $[x,y]$ if and only if $u = y$ but $v \neq x$. Now, if $([x,y],[z,x]) \in O_3$, then $([z,x],[x,y]) \notin O_3$. So, Γ_3 is directed and hence paired with another. It is connected since there exists a path between any two vertices, and has girth 3 since there is a directed path joining the vertices $[1,2]$, $[3,1]$ and $[2,3]$. To show that it is regular where each vertex has indegree $n-2$ and outdegree $n-2$, without loss of generality, there is a directed edge from $[1,2]$ to each of the $n-2$ vertices $[3,1]$, $[4,1], \cdots, [n-1,1]$, and $[n,1]$, and a directed edge from each of the $n-2$ vertices $[2,3]$, $[2,4], \cdots, [2,n-1]$, and $[2,n]$, to $[1,2]$.

Example 5.2.3. If $n = 6$, the suborbital graph Γ_3 corresponding to the suborbit $\triangle_3$ is given as in Figure 5.3 below.

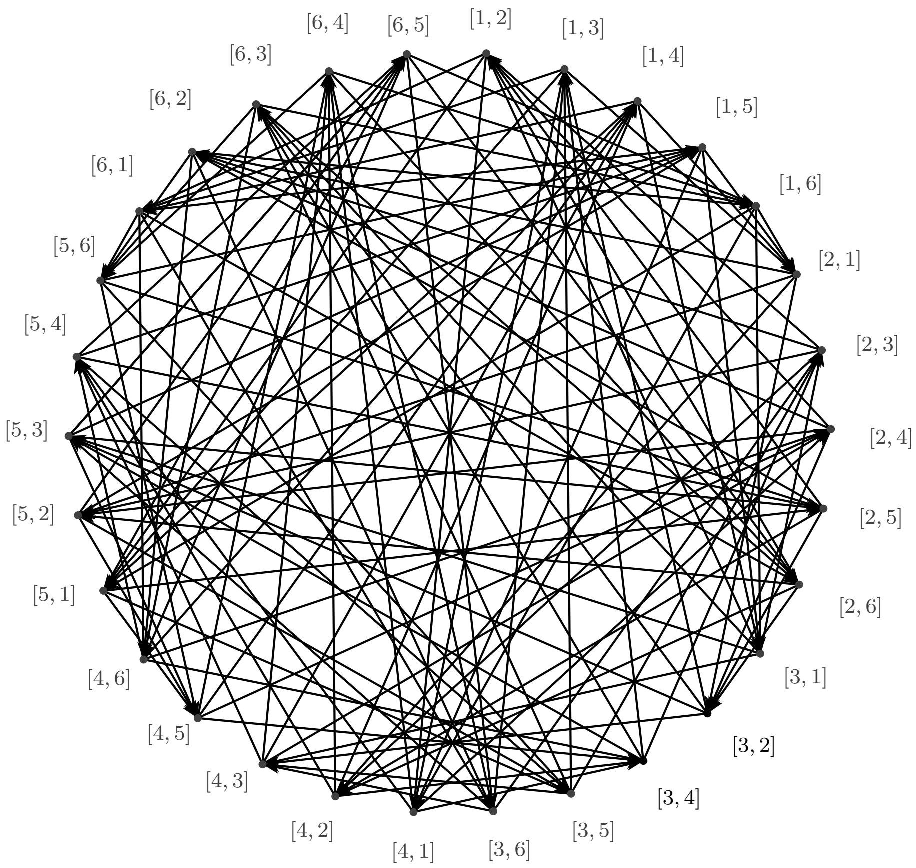

Figure 5.3: Suborbital Graph Γ_3 of A_6 on $X^{[2]}$

(iv) The suborbital O_4 corresponding to the suborbit $\triangle_4$ is

$$O_4 = \{(g[1,2], g[2,3])|g \in G, [2,3] \in \triangle_4\}.$$

So, the corresponding suborbital graph Γ_4 has a directed edge from $[u, v]$ to $[x, y]$ if and only if $v = x$ but $u \neq y$. If $([x, y], [y, z]) \in O_4$, then $([y, z], [x, y]) \notin O_4$. In fact $([y, z], [x, y]) \in O_3$ above. So, Γ_4 is directed and is paired with Γ_3 above and has the same properties as Γ_3, except that the edges are oppositely directed to those of Γ_3.

Example 5.2.4. If $n = 6$, the suborbital graph Γ_4 corresponding to the suborbit $\triangle_4$ is given as in Figure 5.4 below.

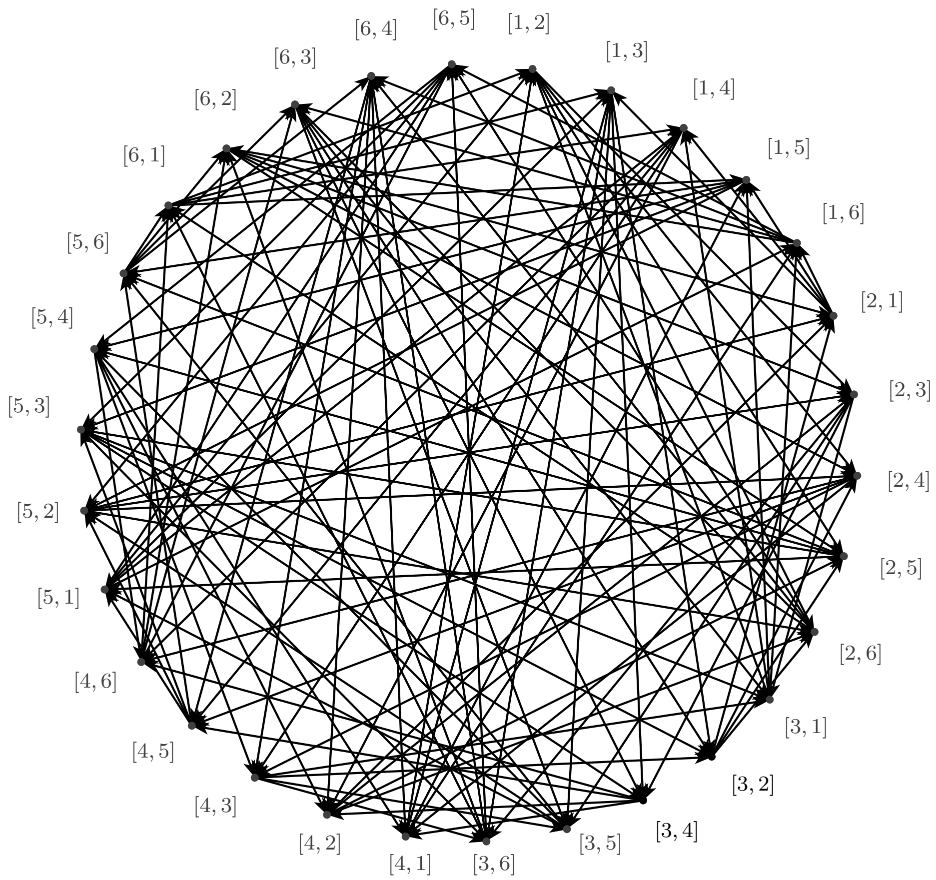

Figure 5.4: Suborbital Graph Γ_4 of A_6 on $X^{[2]}$

(v) The suborbital O_5 corresponding to the suborbit $\triangle_5$ is

$$O_5 = \{(g[1,2], g[3,2]) | g \in G, [3,2] \in \triangle_5\}.$$

So, the corresponding suborbital graph Γ_5 has an edge from $[u,v]$ to $[x,y]$ if and only if $v = y$ and $u \neq x$. Now, if $([x,y],[z,y]) \in O_5$, then $([z,y],[x,y]) \in O_5$. So, Γ_5 is undirected and hence self-paired. In this case, all the properties of Γ_5 are identical to those of Γ_2 above.

Example 5.2.5. If $n = 6$, the suborbital graph Γ_5 corresponding to the suborbit $\triangle_5$ is given as in Figure 5.5 below.

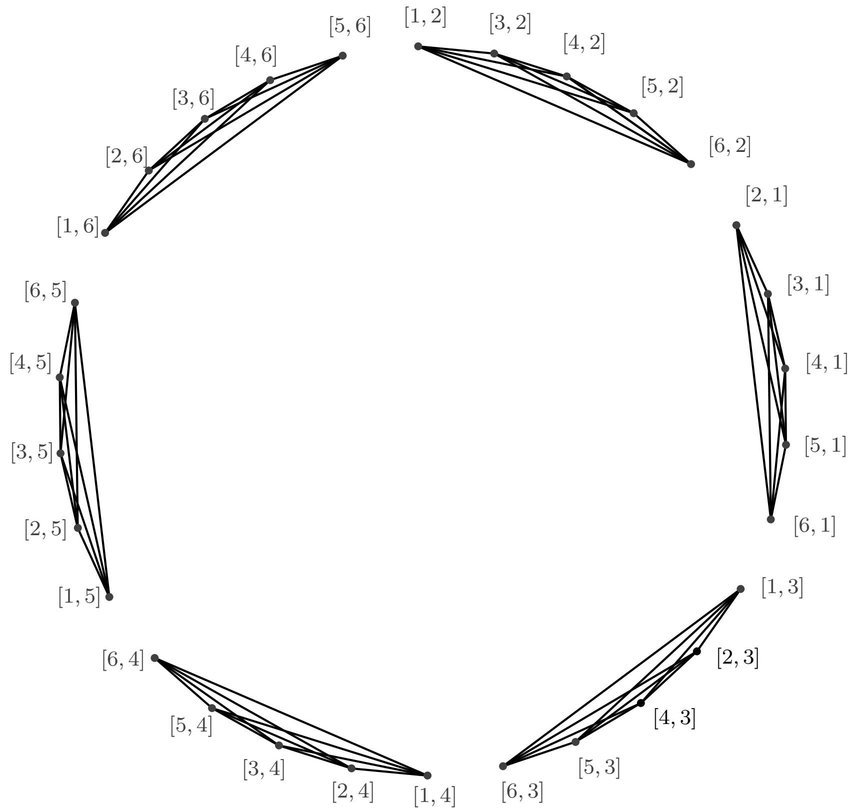

Figure 5.5: Suborbital Graph Γ_5 of A_6 on $X^{[2]}$

(vi) The suborbital O_6 corresponding to the suborbit $\triangle_6$ is

$$O_6 = \{(g[1,2], g[3,4])|g \in G, [3,4] \in \triangle_6\}.$$

The corresponding graph Γ_6 has an edge from $[u, v]$ to $[x, y]$ if and only if $\{u, v\} \cap \{x, y\} = \emptyset$. If $([u, v], [x, y]) \in O_6$, then $([x, y], [u, v]) \in O_6$. So, Γ_6 is undirected and hence self-paired. It is connected since there exists a path between any two vertices of the graph. To show that it is regular of degree $(n-2)(n-3)$, any vertex of the form $[u, v]$ is adjacent to each of the ${}_{n-2}P_2 = \frac{(n-2)!}{(n-4)!} = (n-2)(n-3)$ vertices of the form $[x, y]$ such that $\{u, v\} \cap \{x, y\} = \emptyset$. It has girth 3 if and only if $n \geq 6$ since $[1, 2]$, $[3, 4]$, and $[5, 6]$ are pairwise adjacent if and only if $n \geq 6$.

Example 5.2.6. If $n = 6$, the suborbital graph Γ_6 corresponding to the suborbit $\triangle_6$ is given as in Figure 5.6 below.

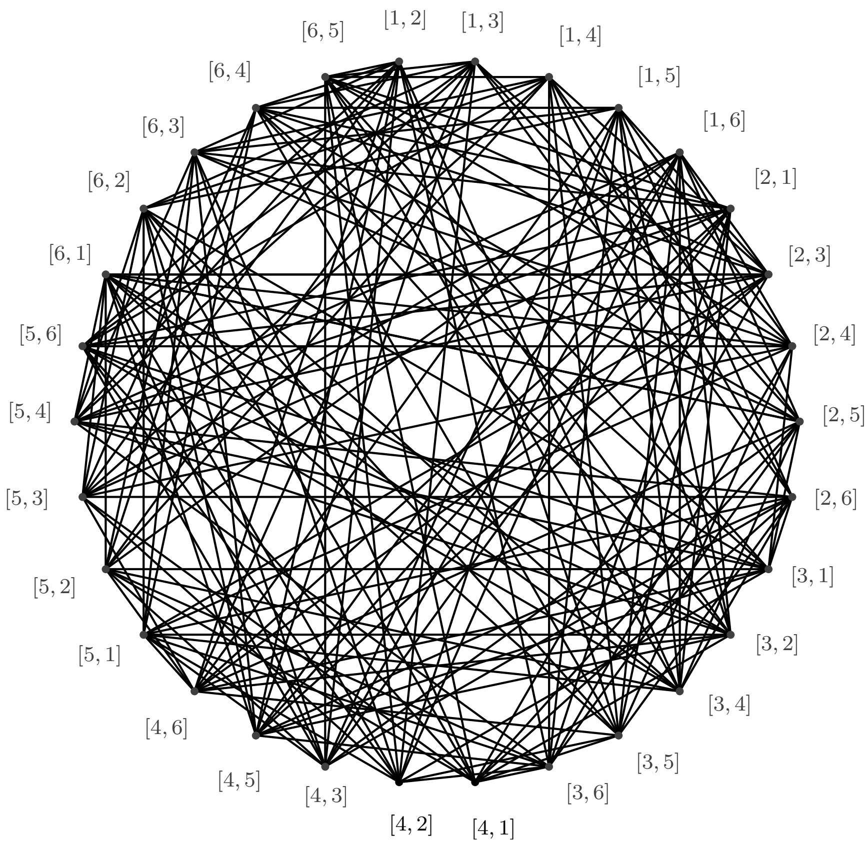

Figure 5.6: Suborbital Graph Γ_6 of A_6 on $X^{[2]}$

5.3 Suborbital Graphs of $\mathbf{A_n}$ on $\mathbf{X^{[r]}}$

5.3.1 Construction of the Suborbital Graphs

Let G act on $X^{[r]}$ and let $[x_1, x_2, \cdots, x_r] \in \triangle$ where $\triangle$ is an orbit of $G_{[1,2,\cdots,r]}$ on $X^{[r]}$. Then the suborbital O corresponding to $\triangle$ is defined by

$$O = \{(g[1, 2, \cdots, r], g[x_1, x_2, \cdots, x_r])|g \in G, [x_1, x_2, \cdots, x_r] \in \triangle\}.$$

The suborbital graph corresponding to the suborbital O has $X^{[r]}$ as its vertex set. Now, suppose $|\{1, 2, \cdots, r\} \cap \{x_1, x_2, \cdots, x_r\}| = k$, $0 \leq k \leq r$ such that the coordinates of $[1, 2, \cdots, r]$ in positions $\alpha_1, \alpha_2, \cdots, \alpha_k$ are respectively identical to the coordinates of $[x_1, x_2, \cdots, x_r]$ in the positions $\beta_1, \beta_2, \cdots, \beta_k$ where $\alpha_i, \beta_j \in \{1, 2, \cdots, r\}$ and $i, j \in \{1, 2, \cdots, k\}$. The graph has an edge from $[y_1, y_2, \cdots, y_r]$ to $[z_1, z_2, \cdots, z_r]$ if and only if $([y_1, y_2, \cdots, y_r], [z_1, z_2, \cdots, z_r])$ is in O, which in turn occurs if and only if $|\{y_1, y_2, \cdots, y_r\} \cap \{z_1, z_2, \cdots, z_r\}| = k$ in such a way that the coordinates of $[y_1, y_2, \cdots, y_r]$ in positions $\alpha_1, \alpha_2, \cdots, \alpha_k$ are respectively identical to the coordinates of $[z_1, z_2, \cdots, z_r]$ in positions $\beta_1, \beta_2, \cdots, \beta_k$.

Example 5.3.1. Let G act on $X^{[4]}$ with $n \geq 10$. Then the ordered quadruple $[5, 1, 2, 7]$ belongs to the suborbit $\triangle_{130} = Orb_{G_{[1,2,3,4]}}[5,1,2,6]$ (find $\triangle_{130}$ in Appendix B). The suborbital O_{130} corresponding to $\triangle_{130}$ is defined by

$$O_{130} = \{(g[1,2,3,4], g[5,1,2,7])|g \in G, [5,1,2,7] \in \triangle_{130}\}.$$

Thus, the graph Γ_{130} corresponding to O_{130} has $X^{[4]}$ as its vertex set. It has an edge from $[x_1, x_2, x_3, x_4]$ to $[y_1, y_2, y_3, y_4]$ if and only if $x_1 = y_2$, $x_2 = y_3$ and $\{x_3, x_4\} \cap \{y_1, y_4\} = \emptyset$. In this case $\triangle_{130}$ is paired with the suborbit $\triangle_{156}$ (see $\triangle_{156}$ in Appendix B, and pairing of the suborbits in Example 3.5.3). Hence, Γ_{130} is paired with the suborbital graph Γ_{156}, which corresponds to $\triangle_{156}$, and each has directed edges.

5.3.2 Properties of the Suborbital Graphs

Theorem 5.3.1. The action of G on $X^{[r]}$ has at least one disconnected non-trivial suborbital graph for all $n \geq r+2$.

Proof. This action is imprimitive, from Theorem 3.3.1 and therefore, by Theorem 1.1.12, the action has a disconnected non-trivial suborbital graph. $\square$

Example 5.3.2. Consider the action of A_n $(n \geq 6)$ on $X^{[2]}$. The suborbital graphs Γ_1, Γ_2 and Γ_5 described in Section 5.2, associated with the action, are all disconnected.

Theorem 5.3.2. Let G act on $X^{[r]}$ and let T_i be the suborbital graph corresponding to a self-paired orbit of $G_{[1,2,\cdots,r]}$ on $X^{[r]}$ whose element has all elements from $\{1, 2, \cdots, r\}$. Then T_i has girth zero, it is regular of degree 1 and is disconnected with $\frac{r!}{2}\binom{n}{r}$ connected components.

Proof. Each vertex of T_i has degree one so that the connected components of T_i are trees with two vertices and one edge (see Figure 5.7 below). Thus, T_i is disconnected, has girth zero and the degree of each vertex is 1. Clearly, the number of connected components in T_i is given by $\frac{|X^{[r]}|}{2} = \frac{1}{2}\left({}_nP_r\right) = \frac{r!}{2}\binom{n}{r}$. $\square$

Figure 5.7: A Connected Component in T_i

Example 5.3.3. A good illustration is the suborbital graph Γ_1 corresponding to the self-paired suborbit $\triangle_1$ of the action of A_6 on $X^{[2]}$ (see Example 5.2.1 above). One can easily check that the number of connected components of Γ_1 is $\frac{2!}{2}\binom{6}{2} = 15$.

Theorem 5.3.3. Let G act on $X^{[r]}$ and let T_j be the suborbital graph corresponding to a paired orbit of $G_{[1,2,\cdots,r]}$ on $X^{[r]}$ whose element has all elements from $\{1, 2, \cdots, r\}$. Then, T_j is disconnected with $\frac{r!}{3}\binom{n}{r}$ connected components and has girth three.

Proof. Each vertex of T_j has indegree 1 and outdegree 1. Construction shows that the connected components of T_j are directed triangles. So, T_j has girth 3 and the number of connected components in T_j is $\frac{|X^{[r]}|}{3} = \frac{1}{3}\left({}_nP_r\right) = \frac{r!}{3}\binom{n}{r}$. $\square$

Example 5.3.4. Let G act on $X^{[3]}$ with $n \geq 8$. The suborbits $\triangle_3$ and $\triangle_4$ (find these suborbits in Appendix A) are paired (see Example 3.5.2). Hence, their corresponding suborbital graphs T_3 and T_4, respectively, are paired. A connected component in T_3 is given in Figure 5.8 below.

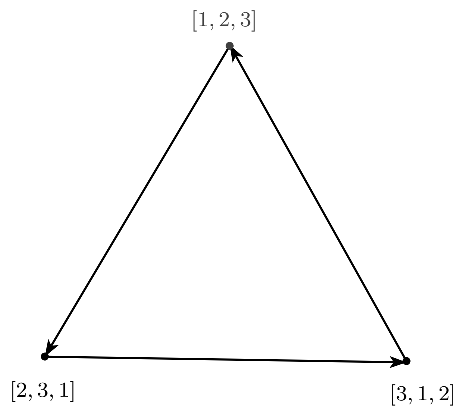

Figure 5.8: A Connected Component in T_3

Theorem 5.3.4. Let G act on $X^{[r]}$ and let T be the suborbital graph corresponding to the self-paired orbit of $G_{[1,2,\cdots,r]}$ on $X^{[r]}$ whose each element contains no element from $\{1, 2, \cdots, r\}$. Then, T is connected and undirected. In addition, T has girth 3 if and only if $n \geq 3r$, and it is regular of degree $(n-r)(n-r-1)\cdots(n-2r+1)$.

Proof. Suppose $\triangle$ is the orbit of $G_{[1,2,\cdots,r]}$ on $X^{[r]}$ whose each element has no element from $\{1, 2, \cdots, r\}$. If $[u_1, u_2, \cdots, u_r] \in \triangle$, then the suborbital corresponding to $\triangle$ is given by

$$O = \{(g[1, 2, \cdots, r], g[u_1, u_2, \cdots, u_r])|g \in G\}.$$

Clearly, $\{1, 2, \cdots, r\} \cap \{u_1, u_2, \cdots, u_r\} = \emptyset$ and the corresponding graph T has an edge from $[v_1, v_2, \cdots, v_r]$ to $[w_1, w_2, \cdots, w_r]$ if and only if $\{v_1, v_2, \cdots, v_r\} \cap \{w_1, w_2, \cdots, w_r\} = \emptyset$.

That the graph is undirected is clear since if $([v_1, v_2, \cdots, v_r], [w_1, w_2, \cdots, w_r]) \in O$, then $([w_1, w_2, \cdots, w_r], [v_1, v_2, \cdots, v_r]) \in O$ also. Now, construction shows that if U and V are distinct vertices of T, there is a path that starts at U and ends at V. Hence, T is connected. Further, the three vertices $[u_1, u_2, \cdots, u_r]$, $[v_1, v_2, \cdots, v_r]$ and $[w_1, w_2, \cdots, w_r]$ will form a cycle if and only if they are $\{u_1, u_2, \cdots, u_r\} \cap \{v_1, v_2, \cdots, v_r\} \cap \{w_1, w_2, \cdots, w_r\} = \emptyset$, which is possible if and only if $n \geq 3r$ (see Figure 5.9 below for a cycle in T). Finally, T is regular of degree $(n-r)(n-r-1)\cdots(n-2r+1)$ since any vertex $[x_1, x_2, \cdots, x_r]$ is adjacent to each of the ${}_{n-r}P_r = \frac{(n-r)!}{(n-2r)!} = (n-r)(n-r-1)\cdots(n-2r+1)$ vertices $[y_1, y_2, \cdots, y_r]$ for which $\{x_1, x_2, \cdots, x_r\} \cap \{y_1, y_2, \cdots, y_r\} = \emptyset$. $\square$

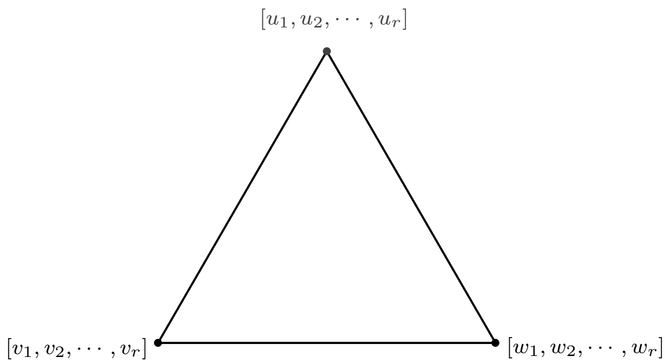

Figure 5.9: A Cycle in T

Example 5.3.5. The suborbital graph Γ_6 seen in Example 5.2.6 above, associated with the action of A_6 on $X^{[2]}$, is one such example.

CHAPTER SIX
SUBORBITAL GRAPHS OF THE ACTION OF $\mathbf{A_n}$ ON UNORDERED SUBSETS

6.1 Introduction

Central to the study of a group action are the properties of the associated combinatorial structures. This chapter sets out to construct the suborbital graphs corresponding to the action of A_n on $X^{(r)}$ and also provide an examination of these graphs with respect to concepts such as directedness, connectedness, number of components in a disconnected graph, vertex degree, and girth. Construction and analysis of the suborbital graphs corresponding to the non-trivial suborbits of the action of A_n on $X^{(2)}$ is handled in Section 6.2. On the other hand, the construction and detailed analysis of suborbital graphs corresponding to more general action of A_n on $X^{(r)}$ is done Section 6.3.

6.2 Suborbital Graphs of $\mathbf{A_n}$ on $\mathbf{X^{(2)}}$

A suborbital graph corresponding to a suborbit of the action of G on $X^{(2)}$ has $X^{(2)}$ as its vertex set. The non-trivial suborbital graphs of G on $X^{(2)}$ are constructed and described as follows:
(i) A suborbital O_1 corresponding to a suborbit $\triangle$ of G on $X^{(2)}$ whose each element contains exactly one of 1 and 2 is

$$O_1 = \{(g\{1,2\}, g\{1,3\})|g \in G, \{1,3\} \in \triangle\}.$$

So, the corresponding suborbital graph has an edge from $\{u,v\}$ to $\{x,y\}$ if and only if $|\{u,v\} \cap \{x,y\}| = 1$. The graph is undirected since if $(\{u,v\},\{x,y\})$ belongs to O_1, so does $(\{x,y\},\{u,v\})$. It is connected as there exists a path between any two distinct vertices, and it has girth 3 because the vertices $\{1,2\}$, $\{1,3\}$ and $\{2,3\}$ are pairwise adjacent. To show that it is regular of degree $\binom{2}{1}\binom{n-2}{1}$, without loss of generality, there is an edge from the vertex $\{1,2\}$ to each of $\{1,3\},\cdots,\{1,n\},\{2,3\},\cdots,\{2,n\}$; the number of edges is $2(n-2) = \binom{2}{1}\binom{n-2}{1}$ in this case.

Example 6.2.1. If A_3 acts on $X^{(2)}$, the paired suborbits $\triangle_1$ and $\triangle_2$ (check pairing of the suborbits in Example 4.5.1) have precisely the same suborbital graph. This graph Φ is undirected and is as given in Figure 6.1 below.

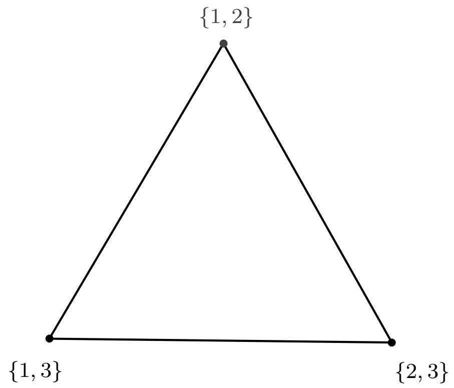

Figure 6.1: A Suborbital Graph Φ of A_3 on $X^{(2)}$

Example 6.2.2. Similarly, the paired suborbits $\triangle_1$ and $\triangle_2$ of A_4 acting on $X^{(2)}$, (see pairing in Example 4.5.1) have precisely the same suborbital graph. This graph Ψ is undirected and is as given in Figure 6.2 below.

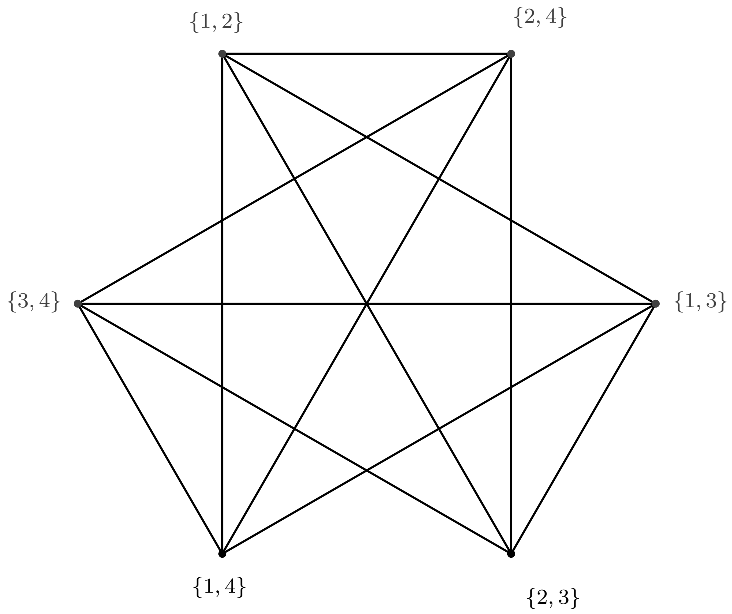

Figure 6.2: A Suborbital Graph Ψ of A_4 on $X^{(2)}$

Example 6.2.3. Suppose A_5 acts on $X^{(2)}$. Then, the suborbital graph Ω corresponding to the suborbit $\triangle_1$ (find $\triangle_1$ in Subsubsection 4.4.1.3) is as shown in Figure 6.3 below.

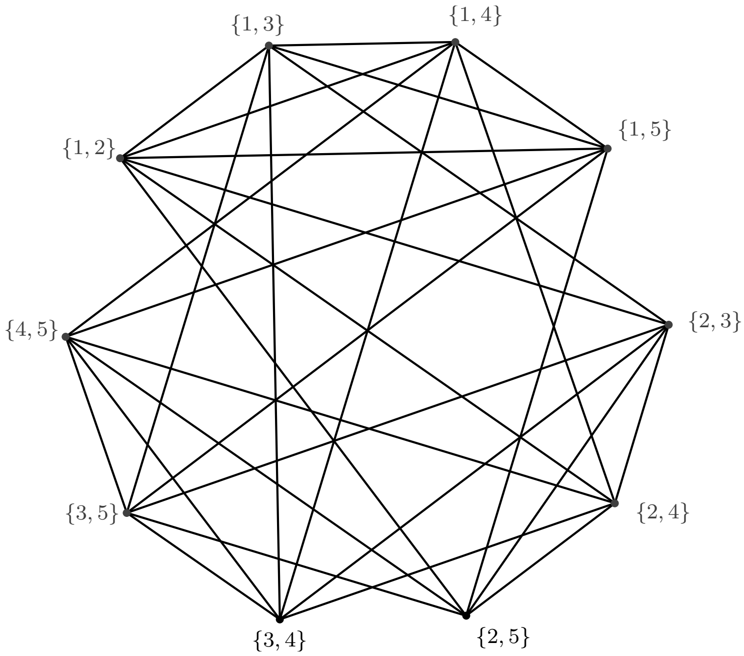

Figure 6.3: Suborbital Graph Ω of A_5 on $X^{(2)}$

(ii) The suborbital O_2 corresponding to the suborbit $\triangle$ of G on $X^{(2)}$ whose each element contains neither of 1 and 2 is

$$O_2 = \{(g\{1,2\}, g\{3,4\}) | g \in G, \{3,4\} \in \triangle\}.$$

In this case, the corresponding suborbital graph has an edge from $\{u,v\}$ to $\{x,y\}$ if and only if $\{u,v\} \cap \{x,y\} = \emptyset$. The graph is undirected since if $(\{u,v\},\{x,y\})$ is in O_2, so does $(\{x,y\},\{u,v\})$. It is regular of degree $\binom{2}{0}\binom{n-2}{2}$ since, without loss of generality, there is an edge from the vertex $\{1,2\}$ to each of the $\binom{n-2}{2} = \binom{2}{0}\binom{n-2}{2}$ vertices of the form $\{x,y\}$, $x,y \in \{3,4,\cdots,n\}$. If $n = 4$, the graph is disconnected and has girth zero (see Figure 6.4 in Example 6.2.4 below). On the other hand if $n \geq 5$, the graph is connected since there is a path between any two distinct vertices, and has girth 5 since $\{1,2\}\{3,4\}\{2,5\}\{1,4\}\{3,5\}\{1,2\}$ is a cycle of shortest length in the graph.

Example 6.2.4. Suppose $G = A_4$ acts on $X^{(2)}$. The suborbital graph $\varGamma_1$ corresponding to the suborbit $\triangle_3$ (see Subsubsection 4.4.1.2 for $\triangle_3$) is as shown in Figure 6.4 below.

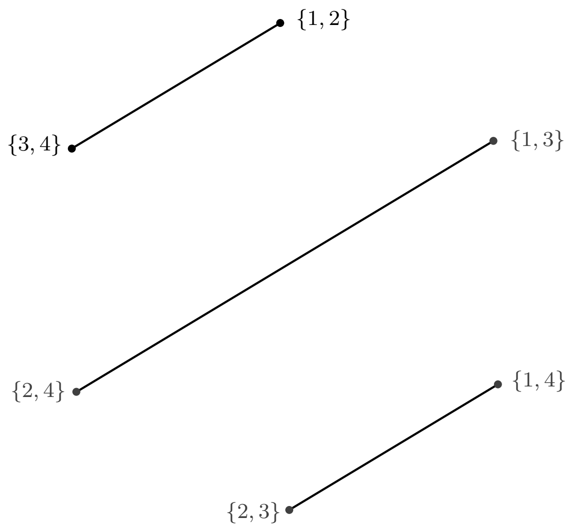

Figure 6.4: Suborbital Graph $\mathit{\Gamma}_1$ of A_4 on $X^{(2)}$

Example 6.2.5. Suppose A_5 acts on $X^{(2)}$. Then, the suborbital graph $\mathit{\Gamma}_2$ corresponding to the suborbit $\triangle_2$ (see Subsubsection 4.4.1.3 for $\triangle_2$) is as shown in Figure 6.5 below.

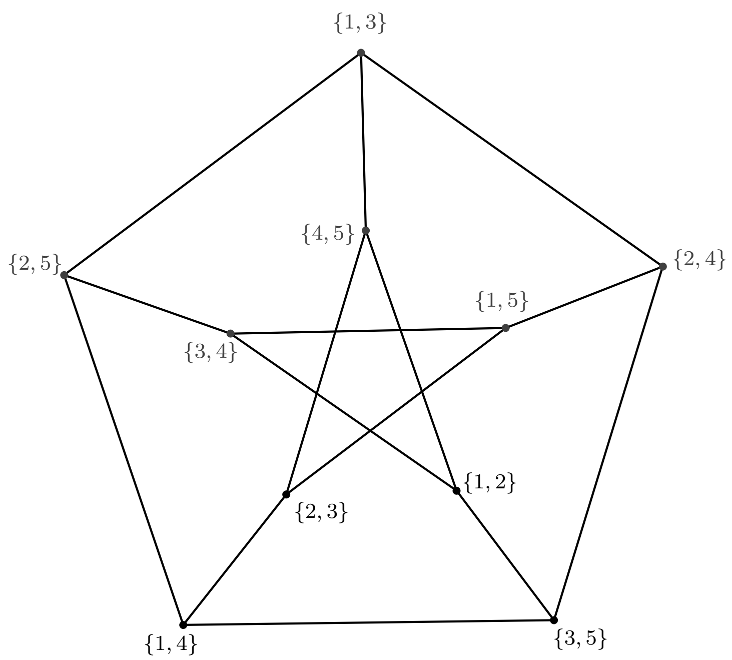

Figure 6.5: Suborbital Graph $\mathit{\Gamma}_2$ of A_5 on $X^{(2)}$

Figure 6.5 above is the famous Petersen Graph.

6.3 Suborbital Graphs of $\mathbf{A_n}$ on $\mathbf{X^{(r)}}$

6.3.1 Construction of the Suborbital Graphs

Let G act on $X^{(r)}$ and let $\triangle$ be a suborbit of G. Then, the suborbital O corresponding to $\triangle$ is given by

$$O = \{(g\{1, 2, \cdots, r\}, g\{x_1, x_2, \cdots, x_r\}) \,|g \in G, \{x_1, x_2, \cdots, x_r\} \in \triangle\} .$$

The suborbital graph Γ corresponding to this suborbital is constructed by taking $X^{(r)}$ as the vertex set. Suppose $|\{1, 2, \cdots, r\} \cap \{x_1, x_2, \cdots, x_r\}| = k$, $0 \leq k \leq r$. Consider vertices $U = \{y_1, y_2, \cdots, y_r\}$ and $V = \{z_1, z_2, \cdots, z_r\}$. Then Γ has an edge from U to V if and only if $(U, V) \in O$, which happens if and only if $|U \cap V| = k$.

Example 6.3.1. Suppose G acts on $X^{(4)}$ with $n \geq 8$. Then $\{2, 3, 5, 6\} \in \triangle_2$ in which case $\triangle_2 = Orb_{G_{\{1,2,\cdots,r\}}}\{1, 2, 5, 6\}$ (see this suborbit in Subsection 4.4.3). The suborbital corresponding to $\triangle_2$ is given by

$$O_2 = \{(g\{1, 2, 3, 4\}, g\{2, 3, 5, 6\}) \,|g \in G, \{2, 3, 5, 6\} \in \triangle_2\},$$

and the suborbital graph Γ_2 corresponding to O_2 has $X^{(4)}$ as the vertex set. Consider the elements $U = \{y_1, y_2, y_3, y_4\}$ and $V = \{z_1, z_2, z_3, z_4\}$ of $X^{(4)}$. Then $(U, V) \in O_2$ if and only if $|U \cap V| = 2$. Accordingly, there exists an edge from U to V. Now, since $\triangle_2$ is self-paired, by Theorem 4.5.1, Γ_2 is self-paired and hence undirected.

6.3.2 Properties of the Suborbital Graphs

Theorem 6.3.1. Every suborbital graph of G acting on $X^{(r)}$ is undirected.

Proof. The trivial suborbital graph is undirected by default. From Examples 6.2.1 and 6.2.2, the suborbital graphs corresponding to the paired suborbits $\triangle_1$ and $\triangle_2$ for the case where $G = A_3$ (or A_4) acts on $X^{(2)}$ are one and the same, and undirected. From Theorem 4.5.1, all the other suborbits are self-paired. Therefore, by Theorem 1.1.11, their corresponding suborbital graphs are self-paired and hence undirected. □

Theorem 6.3.2. The action of G on $X^{(r)}$ has exactly one disconnected non-trivial suborbital graph if $n = 2r$; any other non-trivial suborbital graph of the action is connected.

Proof. Suppose $n = 2r$ so that $X = \{1, 2, \cdots, r, r + 1, r + 2, \cdots, 2r\}$. Now, consider the suborbit $\triangle_r = \{r + 1, r + 2, \cdots, 2r\}$ whose only element contains no element from the set $N = \{1, 2, \cdots, r\}$. Then, the suborbital corresponding to $\triangle_r$ is

$$O_r = \{(g\{1, 2, \cdots, r\}, g\{r + 1, r + 2, \cdots, 2r\}) \,|g \in G\}.$$

The corresponding suborbital graph $\varGamma_r$ has an edge between vertices $U = \{x_1, x_2, \cdots, x_r\}$ and $V = \{y_1, y_2, \cdots, y_r\}$ if and only if $U \cap V = \emptyset$. Since $n = 2r$, U is adjacent only to V, and vise versa, so that a component of $\varGamma_r$ is a tree with two vertices and one edge. Hence $\varGamma_r$ is disconnected with girth zero and the number of connected components is given by $\frac{|X^{(r)}|}{2} = \frac{1}{2}\binom{n}{r}$ in this case. Now, construction shows that there exists a path between any two vertices of any other non-trivial suborbital graph of the action so that it is connected. On the other hand, if $n \neq 2r$, then from Theorem 4.3.1, the action of G on $X^{(r)}$ is primitive and hence, by Theorem 1.1.12, all the non-trivial suborbital graphs associated with this action are connected. $\square$

Example 6.3.2. The suborbital graph $\varGamma_1$ in Example 6.2.4 above is one such disconnected graph.

Theorem 6.3.3. The suborbital graph $\varGamma_{r-i}$ $(i = 0, 1, \cdots, r-1)$ corresponding to the non-trivial suborbit $\triangle_{r-i}$ of G on $X^{(r)}$ whose every element contains exactly i elements from $\{1, 2, \cdots, r\}$ is regular of degree $\binom{r}{i}\binom{n-r}{r-i}$.

Proof. Suppose $\{x_1, x_2, \cdots, x_r\} \in \triangle_{r-i}$. Then the corresponding suborbital is

$$O_{r-i} = \{(g\{1, 2, \cdots, r\}, g\{x_1, x_2, \cdots, x_r\}) \mid g \in G, \{x_1, x_2, \cdots, x_r\} \in \triangle_{r-i}\}.$$

In this case $|\{1, 2, \cdots, r\} \cap \{x_1, x_2, \cdots, x_r\}| = i$. Accordingly, the corresponding suborbital graph $\varGamma_{r-i}$ has $X^{(r)}$ as the vertex set and there exists an edge from vertex $U = \{y_1, y_2, \cdots, y_r\}$ to vertex $V = \{z_1, z_2, \cdots, z_r\}$ if and only if $|U \cap V| = i$. So, the first i coordinates of V are chosen from r elements of X in $\binom{r}{i}$ different ways and the remaining $r - i$ coordinates chosen from the remaining $n - r$ elements of X in $\binom{n-r}{r-i}$ ways. As a result, vertex U is connected to $\binom{r}{i}\binom{n-r}{r-i}$ distinct vertices. Since U is arbitrary, the conclusion is clear. $\square$

Example 6.3.3. If $G = A_5$ acts on $X^{(2)}$, the non-trivial suborbital graphs Ω and $\varGamma_2$ (see Section 6.2) are regular of degree $\binom{2}{1}\binom{3}{1} = 6$ and $\binom{2}{0}\binom{3}{2} = 3$, respectively.

Theorem 6.3.4. Let $\varGamma$ be a suborbital graph of G acting on $X^{(r)}$ where adjacent vertices $U = \{x_1, x_2, \cdots, x_r\}$ and $V = \{y_1, y_2, \cdots, y_r\}$ are such that $|U \cap V| = r - 1$. Then $\varGamma$ has girth 3 .

Proof. Consider the vertices $\{1, 2, \cdots, r-2, r-1, r\}$, $\{1, 2, \cdots, r-2, r-1, r+1\}$ and $\{2, 3, \cdots, r-1, r, r+1\}$ of $\varGamma$. They are pairwise adjacent, by the condition of the theorem, and the conclusion follows. $\square$

Example 6.3.4. If G acts on $X^{(4)}$, then for each $n \geq 5$ there exists a suborbital graph satisfying the condition in Theorem 6.3.4. This graph has a closed path connecting the vertices $\{1, 2, 3, 4\}$, $\{1, 2, 3, 5\}$ and $\{2, 3, 4, 5\}$. In fact this is the suborbital graph corresponding to the suborbit $\triangle_1$ whose each element contains exactly 3 elements from the set $\{1, 2, 3, 4\}$, for each $n \geq 5$ (check Subsection 4.4.3 for the suborbit).

Theorem 6.3.5. Every non-trivial suborbital graph of G on $X^{(r)}$ has girth 3 if $n \geq 3r$.

Proof. Suppose $n \geq 3r$. Let $\mathit{\Gamma}_i$ $(i = 1, 2, \cdots, r)$ be the non-trivial suborbital graph corresponding to a suborbit $\triangle_i$ $(i = 1, 2, \cdots, r)$. Clearly, since $n \geq 3r$ by hypothesis, there exist distinct elements $U = \{u_1, u_2, \cdots, u_r\}$, $V = \{v_1, v_2, \cdots, v_r\}$ and $W = \{w_1, w_2, \cdots, w_r\}$ in $X^{(r)}$ such that $|U \cap V| = |U \cap W| = |V \cap W| = k$, $k = 0, 1, \cdots, r-1$. Hence U, V and W are pairwise adjacent in $\mathit{\Gamma}_i$, so that $\mathit{\Gamma}_i$ has a cycle of length 3. $\square$

Example 6.3.5. The action of A_6 on $X^{(2)}$ has two non-trivial suborbits $\triangle_1$ and $\triangle_2$ (see Subsubsection 4.4.1.4). From the argument given in Subsection 6.3.1 above, the suborbital graph corresponding to $\triangle_1$ has a cycle joining the vertices $\{1, 2\}$, $\{1, 3\}$ and $\{1, 4\}$. Similarly, the suborbital graph corresponding to $\triangle_2$ has pairwise adjacent vertices $\{1, 2\}$, $\{3, 4\}$ and $\{5, 6\}$.

CHAPTER SEVEN

CYCLE INDEX OF $\mathbf{S_n}$ AS A SEMIDIRECT PRODUCT OF $\mathbf{A_n}$ BY $\mathbf{C_2}$

7.1 Introduction

The idea of the cycle index formula of a group action is a significant one. The current chapter is devoted for the derivation of the cycle index formula of the symmetric group S_n, explicitly in terms of the cycle index formula of A_n and that of C_2, where S_n is a semidirect product of A_n by C_2. The expression of cycle index of S_n in terms of cycle indices of A_n and C_2 for each case where $3 \leq n \leq 7$ is done in Section 7.2. Elsewhere, the derivation of the cycle index of S_n in terms of the cycle index of A_n and that of C_2 for any $n \geq 3$ takes place in Section 7.3. Throughout the chapter, the notation $Z_{1,X}$ shall be reserved to mean the cycle index of the trivial group acting on X. On the other hand, the notations G, H, and K shall be used as defined in the respective contexts but not necessarily as defined in other contexts before.

7.2 Cycle Index of $\mathbf{S_n}$ as a Semidirect Product of $\mathbf{A_n}$ by $\mathbf{C_2}$ for $\mathbf{3 \leq n \leq 7}$

7.2.1 Cycle Index of $\mathbf{S_3}$ as a Semidirect Product of $\mathbf{A_3}$ by $\mathbf{C_2}$

Consider the set $X = \{1, 2, 3\}$ and the groups $G = \{1, (123), (132), (12), (13), (23)\} = S_3$, $H = \{1, (123), (132)\} = A_3$ and $K = \{1, (12)\} = C_2$. By Equation 1.1.1, the cycle index polynomials of G, H, and K acting on X are $Z_{G,X} = \frac{1}{6}\{t_1^3+2t_3+3t_1t_2\}$, $Z_{H,X} = \frac{1}{3}\{t_1^3+2t_3\}$, and $Z_{K,X} = \frac{1}{2}\{t_1^3 + t_1t_2\}$, respectively. Now, on rewriting,

$$\begin{aligned} Z_{G,X} &= \frac{1}{6}\left\{t_1^3 + 2t_3\right\} + \frac{1}{6}\left\{3t_1t_2\right\} \\ &= \frac{1}{2}\frac{1}{3}\left\{t_1^3 + 2t_3\right\} + \frac{1}{2}\left\{t_1^3 + t_1t_2\right\} - \frac{1}{2}t_1^3 \\ &= \frac{1}{2}Z_{H,X} + Z_{K,X} - \frac{1}{2}Z_{1,X}. \end{aligned}$$

This formula was originally obtained by (Kamuti, 2004) but using a different method; this is as expected since G is Frobenius (see Example 1.1.5). The formula can further be rewritten as

$$\begin{aligned} Z_{G,X} &= \frac{1}{2}Z_{H,X} + \frac{2!}{3!}\frac{3!}{2!1!}Z_{K,X} - \frac{1}{3!}\frac{3!}{2!1!}Z_{1,X} \\ &= \frac{1}{2}Z_{H,X} + \frac{2}{3!}\binom{3}{2}Z_{K,X} - \frac{1}{3!}\binom{3}{2}Z_{1,X}. \end{aligned}$$

In this case, all the transpositions (odd permutations) in G are conjugate to the transposition (12) in K, by Theorem 1.1.14.

7.2.2 Cycle Index of $\mathrm{S_4}$ as a Semidirect Product of $\mathrm{A_4}$ by $\mathrm{C_2}$

Suppose $X = \{1, 2, 3, 4\}$ so that $G = S_4$ (see Example 1.1.6 for the elements of G),

$$\begin{aligned} H &= \{1, (123), (132), (124), (142), (134), (143), (234), \\ &\qquad (243), (12)(34), (13)(24), (14)(23)\} \\ &= A_4 \end{aligned}$$

and

$$K = \{1, (12)\} = C_2.$$

By Equation 1.1.1, the corresponding cycle index polynomials are

$$Z_{G,X} = \frac{1}{24}\{t_1^4 + 8t_1t_3 + 3t_2^2 + 6t_1^2t_2 + 6t_4\},$$

$$Z_{H,X} = \frac{1}{12}\{t_1^4 + 8t_1t_3 + 3t_2^2\},$$

and

$$Z_{K,X} = \frac{1}{2}\{t_1^4 + t_1^2t_2\}$$

respectively. Now, $Z_{G,X}$ can be rewritten as

$$\begin{aligned} Z_{G,X} &= \frac{1}{24}\left\{t_1^4 + 8t_1t_3 + 3t_2^2\right\} + \frac{1}{24}\left\{6t_1^2t_2 + 6t_4\right\} \\ &= \frac{1}{2}\frac{1}{12}\left\{t_1^4 + 8t_1t_3 + 3t_2^2\right\} \\ &\qquad + \frac{1}{2}\frac{1}{2}\left\{t_1^4 + t_1^2t_2\right\} - \frac{1}{4}t_1^4 + \frac{1}{24}\left\{6t_4\right\} \\ &= \frac{1}{2}Z_{H,X} + \frac{1}{2}Z_{K,X} - \frac{1}{4}Z_{1,X} + \frac{1}{4}t_4 \\ &= \frac{1}{2}Z_{H,X} + \frac{2!}{4!}\frac{4!}{2!2!}Z_{K,X} - \frac{1}{4!}\frac{4!}{2!2!}Z_{1,X} + \frac{1}{4}t_4 \\ &= \frac{1}{2}Z_{H,X} + \frac{2}{4!}\binom{4}{2}Z_{K,X} - \frac{1}{4!}\binom{4}{2}Z_{1,X} + \frac{1}{4}t_4. \end{aligned}$$

In this case, the extra term $\frac{1}{4}t_4$ is the contribution, to $Z_{G,X}$, of the six odd permutations of X that are not transpositions. These permutations are not in a conjugate of K but all of them form a conjugacy class in G, by Theorem 1.1.14.

7.2.3 Cycle Index of $\mathrm{S_5}$ as a Semidirect Product of $\mathrm{A_5}$ by $\mathrm{C_2}$

Let $X = \{1, 2, 3, 4, 5\}$, $G = S_5$, $H = A_5$ and $K = < (12) > = C_2$. The cycle index of K can easily be calculated since its elements and their corresponding monomials can be listed down without much effort. On the other hand, Table 7.1 below is essential in the calculation of the cycle indices of G and H since it is quite cumbersome listing down the elements of each.

Table 7.1: Monomials of Elements of S_5

Permutation Type	Cycle Type	Corresponding Monomial	Corresponding Number of Elements in S_5
$(a)(b)(c)(d)(e)$	$(5,0,0,0,0)$	t_1^5	1
$(a)(bc)(de)$	$(1,2,0,0,0)$	$t_1t_2^2$	15
$(a)(b)(cde)$	$(2,0,1,0,0)$	$t_1^2t_3$	20
$(abcde)$	$(0,0,0,0,1)$	t_5	24
$(a)(b)(c)(de)$	$(3,1,0,0,0)$	$t_1^3t_2$	10
$(ab)(cde)$	$(0,1,1,0,0)$	t_2t_3	20
$(a)(bcde)$	$(1,0,0,1,0)$	t_1t_4	30
Total			$120 = \lvert S_5\rvert$

Now, in a manner analogous to the calculation of $Z_{G,X}$ in Example 1.1.6,

$$Z_{G,X} = \frac{1}{120}\{t_1^5 + 15t_1t_2^2 + 20t_1^2t_3 + 24t_5 + 10t_1^3t_2 + 20t_2t_3 + 30t_1t_4\},$$

and

$$Z_{H,X} = \frac{1}{60}\{t_1^5 + 15t_1t_2^2 + 20t_1^2t_3 + 24t_5\},$$

while

$$Z_{K,X} = \frac{1}{2}\{t_1^5 + t_1^3t_2\}.$$

Just like in Subsections 7.2.1 and 7.2.2, $Z_{G,X}$ can be rewritten as

$$\begin{aligned}
Z_{G,X} &= \frac{1}{120}\left\{t_1^5 + 15t_1t_2^2 + 20t_1^2t_3 + 24t_5\right\} + \frac{1}{120}\left\{10t_1^3t_2 + 20t_2t_3 + 30t_1t_4\right\} \\
&= \frac{1}{2}\frac{1}{60}\left\{t_1^5 + 15t_1t_2^2 + 20t_1^2t_3 + 24t_5\right\} + \frac{1}{6}\frac{1}{2}\left\{t_1^5 + t_1^3t_2\right\} \\
&\qquad -\frac{1}{12}t_1^5 + \frac{1}{120}\{20t_2t_3 + 30t_1t_4\} \\
&= \frac{1}{2}Z_{H,X} + \frac{1}{6}Z_{K,X} - \frac{1}{12}Z_{1,X} + \frac{1}{5!}\{20t_2t_3 + 30t_1t_4\} \\
&= \frac{1}{2}Z_{H,X} + \frac{2!}{5!}\frac{5!}{2!3!}Z_{K,X} - \frac{1}{5!}\frac{5!}{2!3!}Z_{1,X} + \frac{1}{5!}\{20t_2t_3 + 30t_1t_4\} \\
&= \frac{1}{2}Z_{H,X} + \frac{2}{5!}\binom{5}{2}Z_{K,X} - \frac{1}{5!}\binom{5}{2}Z_{1,X} + \frac{1}{5!}\{20t_2t_3 + 30t_1t_4\}.
\end{aligned}$$

The extra term $\frac{1}{5!}\{20t_2t_3 + 30t_1t_4\}$ in this case is the contribution, to $Z_{G,X}$, of the 50 odd permutations of X that are not transpositions. By Theorem 1.1.14, the permutations are not in a conjugate of K but form some two conjugacy classes in G; one consisting of 20 elements and the other 30.

7.2.4 Cycle Index of $\mathbf{S_6}$ as a Semidirect Product of $\mathbf{A_6}$ by $\mathbf{C_2}$

Suppose $X = \{1, 2, 3, 4, 5, 6\}$, $G = S_6$, $H = A_6$ and $K = <(12)> = C_2$. Just like in Subsection 7.2.3, Table 7.2 below is vital in the calculation of $Z_{G,X}$ and $Z_{H,X}$.

Table 7.2: Monomials of Elements of S_6

Permutation Type	Cycle Type	Corresponding Monomial	Corresponding Number of Elements in S_6
$(a)(b)(c)(d)(e)(f)$	$(6,0,0,0,0,0)$	t_1^6	1
$(a)(b)(cd)(ef)$	$(2,2,0,0,0,0)$	$t_1^2t_2^2$	45
$(a)(b)(c)(def)$	$(3,0,1,0,0,0)$	$t_1^3t_3$	40
$(abc)(def)$	$(0,0,2,0,0,0)$	t_3^2	40
$(a)(bcdef)$	$(1,0,0,0,1,0)$	t_1t_5	144
$(ab)(cdef)$	$(0,1,0,1,0,0)$	t_2t_4	90
$(a)(b)(c)(d)(ef)$	$(4,1,0,0,0,0)$	$t_1^4t_2$	15
$(a)(bc)(def)$	$(1,1,1,0,0,0)$	$t_1t_2t_3$	120
$(ab)(cd)(ef)$	$(0,3,0,0,0,0)$	t_2^3	15
$(a)(b)(cdef)$	$(2,0,0,1,0,0)$	$t_1^2t_4$	90
$(abcdef)$	$(0,0,0,0,0,1)$	t_6	120
Total			$720 = \lvert S_6\rvert$

By Equation 1.1.1,

$$\begin{aligned} Z_{G,X} &= \frac{1}{720}\{t_1^6 + 45t_1^2t_2^2 + 40t_1^3t_3 + 40t_3^2 + 144t_1t_5 + 90t_2t_4 \\ &\qquad +15t_1^4t_2 + 120t_1t_2t_3 + 15t_2^3 + 90t_1^2t_4 + 120t_6\}, \end{aligned}$$

$$Z_{H,X} = \frac{1}{360}\{t_1^6 + 45t_1^2t_2^2 + 40t_1^3t_3 + 40t_3^2 + 144t_1t_5 + 90t_2t_4\}$$

and

$$Z_{K,X} = \frac{1}{2}\{t_1^6 + t_1^4t_2\}.$$

Now, $Z_{G,X}$ can be rewritten as

$$\begin{aligned}
Z_{G,X} &= \frac{1}{720}\left\{t_1^6+45t_1^2t_2^2+40t_1^3t_3+40t_3^2+144t_1t_5+90t_2t_4\right\} \\
&\qquad\qquad +\frac{1}{720}\{15t_1^4t_2+120t_1t_2t_3+15t_2^3+90t_1^2t_4+120t_6\} \\
&= \frac{1}{2}\frac{1}{360}\left\{t_1^6+45t_1^2t_2^2+40t_1^3t_3+40t_3^2+144t_1t_5+90t_2t_4\right\} \\
&\quad +\frac{1}{24}\frac{1}{2}\left\{t_1^6+t_1^4t_2\right\}-\frac{1}{48}t_1^6+\frac{1}{720}\left\{120t_1t_2t_3+15t_2^3+90t_1^2t_4+120t_6\right\} \\
&= \frac{1}{2}Z_{H,X}+\frac{1}{24}Z_{K,X}-\frac{1}{48}Z_{1,X}+\frac{1}{6!}\left\{120t_1t_2t_3+15t_2^3+90t_1^2t_4+120t_6\right\} \\
&= \frac{1}{2}Z_{H,X}+\frac{2!}{6!}\frac{6!}{2!4!}Z_{K,X}-\frac{1}{6!}\frac{6!}{2!4!}Z_{1,X} \\
&\qquad\qquad +\frac{1}{6!}\left\{120t_1t_2t_3+15t_2^3+90t_1^2t_4+120t_6\right\} \\
&= \frac{1}{2}Z_{H,X}+\frac{2}{6!}\binom{6}{2}Z_{K,X}-\frac{1}{6!}\binom{6}{2}Z_{1,X} \\
&\qquad\qquad +\frac{1}{6!}\left\{120t_1t_2t_3+15t_2^3+90t_1^2t_4+120t_6\right\}.
\end{aligned}$$

The extra term $\frac{1}{6!}\left\{120t_1t_2t_3+15t_2^3+90t_1^2t_4+120t_6\right\}$ in this case is the contribution, to $Z_{G,X}$, of the 345 odd permutations of X that are not transpositions. These permutations are not in a conjugate of K but form some four conjugacy classes in G.

7.2.5 Cycle Index of $\mathrm{S_7}$ as a Semidirect Product of $\mathrm{A_7}$ by $\mathrm{C_2}$

Take $X=\{1,2,3,4,5,6,7\}$, $G=S_7$, $H=A_7$ and $K=<(12)>=C_2$. While it is easy to find $Z_{K,X}$, Table 7.3 below becomes absolutely necessary in finding $Z_{G,X}$ and $Z_{H,X}$.

Table 7.3: Monomials of Elements of S_7

Permutation Type	Cycle Type	Corresponding Monomial	Corresponding Number of Elements in S_7
$(a)(b)(c)(d)(e)(f)(g)$	$(7,0,0,0,0,0,0)$	t_1^7	1
$(a)(b)(c)(de)(fg)$	$(3,2,0,0,0,0,0)$	$t_1^3t_2^2$	105
$(a)(b)(c)(d)(efg)$	$(4,0,1,0,0,0,0)$	$t_1^4t_3$	70
$(ab)(cd)(efg)$	$(0,2,1,0,0,0,0)$	$t_2^2t_3$	210
$(a)(bcd)(efg)$	$(1,0,2,0,0,0,0)$	$t_1t_3^2$	280
$(a)(bc)(defg)$	$(1,1,0,1,0,0,0)$	$t_1t_2t_4$	630
$(a)(b)(cdefg)$	$(2,0,0,0,1,0,0)$	$t_1^2t_5$	504
$(abcdefg)$	$(0,0,0,0,0,0,1)$	t_7	720
$(a)(b)(c)(d)(e)(fg)$	$(5,1,0,0,0,0,0)$	$t_1^5t_2$	21
$(a)(bc)(de)(fg)$	$(1,3,0,0,0,0,0)$	$t_1t_2^3$	105
$(a)(b)(cd)(efg)$	$(2,1,1,0,0,0,0)$	$t_1^2t_2t_3$	420
$(a)(b)(c)(defg)$	$(3,0,0,1,0,0,0)$	$t_1^3t_4$	210
$(abc)(defg)$	$(0,0,1,1,0,0,0)$	t_3t_4	420
$(ab)(cdefg)$	$(0,1,0,0,1,0,0)$	t_2t_5	504
$(a)(bcdefg)$	$(1,0,0,0,0,1,0)$	t_1t_6	840
Total			$5040 = \|S_7\|$

By Equation 1.1.1,

$$Z_{G,X} = \frac{1}{5040}\{t_1^7 + 105t_1^3t_2^2 + 70t_1^4t_3 + 210t_2^2t_3 + 280t_1t_3^2 + 630t_1t_2t_4 + 504t_1^2t_5 + 720t_7 + 21t_1^5t_2 + 105t_1t_2^3 + 420t_1^2t_2t_3 + 210t_1^3t_4 + 420t_3t_4 + 504t_2t_5 + 840t_1t_6\},$$

$$Z_{H,X} = \frac{1}{2520}\{t_1^7 + 105t_1^3t_2^2 + 70t_1^4t_3 + 210t_2^2t_3 + 280t_1t_3^2 + 630t_1t_2t_4 + 504t_1^2t_5 + 720t_7\}$$

and

$$Z_{K,X} = \frac{1}{2}\{t_1^7 + t_1^5t_2\}.$$

Thus, $Z_{G,X}$ can be rewritten

$$\begin{aligned}
Z_{G,X} &= \frac{1}{5040}\{t_1^7 + 105t_1^3t_2^2 + 70t_1^4t_3 + 210t_2^2t_3 + 280t_1t_3^2 \\
&\qquad +630t_1t_2t_4 + 504t_1^2t_5 + 720t_7\} + \frac{1}{5040}\{21t_1^5t_2 + 105t_1t_2^3 \\
&\qquad +420t_1^2t_2t_3 + 210t_1^3t_4 + 420t_3t_4 + 504t_2t_5 + 840t_1t_6\} \\
&= \frac{1}{2}\frac{1}{2520}\{t_1^7 + 105t_1^3t_2^2 + 70t_1^4t_3 + 210t_2^2t_3 + 280t_1t_3^2 \\
&\qquad +630t_1t_2t_4 + 504t_1^2t_5 + 720t_7\} + \frac{1}{120}\frac{1}{2}\{t_1^7 + t_1^5t_2\} \\
&\qquad -\frac{1}{240}t_1^7 + \frac{1}{5040}\{105t_1t_2^3 + 420t_1^2t_2t_3 + 210t_1^3t_4 \\
&\qquad +420t_3t_4 + 504t_2t_5 + 840t_1t_6\} \\
&= \frac{1}{2}Z_{H,X} + \frac{1}{120}Z_{K,X} - \frac{1}{240}Z_{1,X} \\
&\qquad +\frac{1}{7!}\{105t_1t_2^3 + 420t_1^2t_2t_3 + 210t_1^3t_4 + 420t_3t_4 + 504t_2t_5 + 840t_1t_6\} \\
&= \frac{1}{2}Z_{H,X} + \frac{2!}{7!}\frac{7!}{2!5!}Z_{K,X} - \frac{1}{7!}\frac{7!}{2!5!}Z_{1,X} \\
&\qquad +\frac{1}{7!}\{105t_1t_2^3 + 420t_1^2t_2t_3 + 210t_1^3t_4 + 420t_3t_4 + 504t_2t_5 + 840t_1t_6\} \\
&= \frac{1}{2}Z_{H,X} + \frac{2}{7!}\binom{7}{2}Z_{K,X} - \frac{1}{7!}\binom{7}{2}Z_{1,X} \\
&\qquad +\frac{1}{7!}\{105t_1t_2^3 + 420t_1^2t_2t_3 + 210t_1^3t_4 + 420t_3t_4 + 504t_2t_5 + 840t_1t_6\}.
\end{aligned}$$

The extra term $\frac{1}{7!}\{105t_1t_2^3 + 420t_1^2t_2t_3 + 210t_1^3t_4 + 420t_3t_4 + 504t_2t_5 + 840t_1t_6\}$ in this case is the contribution to $Z_{G,X}$ of the 2499 odd permutations of X that are not transpositions. These permutations are not in a conjugate of K but form some six conjugacy classes in G, of 105, 420, 210, 420, 504, and 840 elements respectively.

7.3 Cycle Index of $\mathbf{S_n}$ as a Semidirect Product of $\mathbf{A_n}$ by $\mathbf{C_2}$

Theorem 7.3.1. Let $X = \{1, 2, \cdots, n\}$ so that $G = S_n$, $H = A_n$ and $K = C_2$. Suppose Ω is the set of the odd permutations of X that are not transpositions. Then,

$$Z_{G,X} = \frac{1}{2}Z_{H,X} + \frac{2}{n!}\binom{n}{2}Z_{K,X} - \frac{1}{n!}\binom{n}{2}Z_{1,X} + \frac{1}{n!}\sum_{g\in\Omega}\{mon(g)\}.$$

Proof. The cycle type of the identity permutation, which is even, is $(n, 0, 0, \cdots, 0)$ while that of a transposition (ab), an odd permutation , is $(n-2, 1, 0, 0, \cdots, 0)$. So, by definition, $mon(1) = t_1^n$ and $mon(ab) = t_1^{n-2}t_2$. Now, suppose p is the sum of monomials of the non-trivial even permutations of X. Then, by Equation 1.1.1, $Z_{H,X} = \frac{2}{n!}\{t_1^n + p\}$. Similarly,

$Z_{K,X} = \frac{1}{2}\left\{t_1^n + t_1^{n-2}t_2\right\}$. The conjugacy class of (ab) has $\frac{n!}{(n-2)!1^{n-2}1!2^1} = \frac{n!}{2!(n-2)!} = \binom{n}{2}$ elements, by Theorem 1.1.14. Finally, suppose Ω is the set of the odd permutations of X that are not transpositions (i.e., Ω consists of all odd permutations of X that are not conjugate to the transposition in K, by Theorem 1.1.14). Then, from Equation 1.1.1,

$$\begin{aligned}
Z_{G,X} &= \frac{1}{n!}\sum_{g\in G}\{mon(g)\} \\
&= \frac{1}{n!}\sum_{g\in H}\{mon(g)\} + \frac{1}{n!}\sum_{g\in G\setminus H}\{mon(g)\} \\
&= \frac{1}{n!}\{t_1^n + p\} + \frac{1}{n!}\left\{\binom{n}{2}t_1^{n-2}t_2 + \sum_{g\in\Omega}\{mon(g)\}\right\} \\
&= \frac{1}{2}\left[\frac{2}{n!}\{t_1^n + p\}\right] + \frac{1}{n!}\binom{n}{2}t_1^{n-2}t_2 + \frac{1}{n!}\sum_{g\in\Omega}\{mon(g)\} \\
&= \frac{1}{2}\left[\frac{2}{n!}\{t_1^n + p\}\right] + \frac{2}{n!}\binom{n}{2}\left[\frac{1}{2}\left\{t_1^n + t_1^{n-2}t_2\right\}\right] \\
&\qquad -\frac{1}{n!}\binom{n}{2}t_1^n + \frac{1}{n!}\sum_{g\in\Omega}\{mon(g)\} \\
&= \frac{1}{2}Z_{H,X} + \frac{2}{n!}\binom{n}{2}Z_{K,X} - \frac{1}{n!}\binom{n}{2}Z_{1,X} + \frac{1}{n!}\sum_{g\in\Omega}\{mon(g)\}.
\end{aligned}$$

□

CHAPTER EIGHT
SUMMARY, CONCLUSIONS AND RECOMMENDATIONS

8.1 Introduction

This chapter provides a summary of the research findings in Chapters 3 through 7 and then draws conclusions from these findings relative to the objectives of the study as outlined in Chapter 1. It finally gives recommendations for some contexts that may attract interest for research in the future.

8.2 Summary

This thesis was set out to investigate transitivity, primitivity, ranks, subdegrees and properties of the suborbital graphs of the actions of the alternating group A_n on $X^{[r]}$ and $X^{(r)}$, respectively the ordered and unordered r-element subsets of $X = \{1, 2, \cdots, n\}$. It also aimed at expressing the cycle index polynomial of the semidirect product $S_n = A_n \rtimes C_2$ explicitly in terms of the cycle index polynomial of A_n and that of C_2.

In Chapter 3, it was shown that for a fixed integer r $(r \geq 2)$ the action of A_n on $X^{[r]}$ is both transitive and imprimitive if and only if $n \geq r + 2$. On the other hand, the rank was found to be a constant integer $(r!)^2 \sum\limits_{i=0}^{r} \frac{1}{(i!)^2(r-i)!}$ whenever $n \geq 2(r + 1)$. The subdegrees of the action were calculated and arranged according to their increasing order of magnitude. Also, some necessary and sufficient conditions for a suborbit to be self-paired or paired with another, as well as a formula for finding the number self-paired suborbits, were determned.

Next, in Chapter 4, it was proved that the action of A_n on $X^{(r)}$ for a fixed integer r $(r \geq 2)$ is transitive if $n \geq r + 1$. Also, it was shown that the action is imprimitive if and only if $n = 2r$. The rank of the action was established to be a constant $r + 1$ if and only if $n \geq 2r$. The subdegrees of the action were also calculated and arranged in increasing order of magnitude. The suborbits of this action, except for some few specified cases, were shown to be self-paired.

Then, Chapter 5 focused on the suborbital graphs associated with the action of A_n on $X^{[r]}$. The graphs were constructed and their properties analysed. In this regard, it was found that the graphs corresponding to the self-paired suborbits whose respective elements have all elements from the set $N = \{1, 2, \cdots, r\}$ are disconnected with girth zero. Also, the graphs corresponding to the paired suborbits whose respective elements contain exactly r elements from N are disconnected with girth three. Moreover, the graph corresponding to the suborbit whose every element contains no element from N is connected, undirected, has girth three if and only if $n \geq 3r$ and is regular of degree $(n - r)(n - r - 1) \cdots (n - 2r + 1)$.

Further, Chapter 6 was concerned with the construction and analysis of the suborbital graphs associated with the action of A_n on $X^{(r)}$. It was found that all the suborbital graphs are undirected. Also, it was found that the action has exactly one disconnected graph if and only if $n = 2r$. In addition, it was established that if any two adjacent vertices of a graph have i $(i = 0, 1, \cdots, r-1)$ elements in common, the graph is regular of degree $\binom{r}{i}\binom{n-r}{r-i}$. Finally, it was proved that if a graph of the action is such that any two adjacent vertices have $r-1$ elements in common, or the graph corresponds to the action of A_n on $X^{(r)}$ with $n \geq 3r$, then it has girth three.

Lastly, in Chapter 7, it was established that if $G = S_n$, $H = A_n$, $K = C_2$, and Ω the set of all odd permutations of X that are not transpositions (i.e., the elements of Ω are not conjugate to the transposition generating K), then the cycle index polynomial of the group G is given explicitly in terms of the cycle index polynomials of its subgroups H and K as $Z_{G,X} = \frac{1}{2}Z_{H,X} + \frac{2}{n!}\binom{n}{2}Z_{K,X} - \frac{1}{n!}\binom{n}{2}Z_{1,X} + \frac{1}{n!}\sum_{g\in\Omega}\{mon(g)\}$.

8.3 Conclusions

It is evident from Section 8.2 that this study has accomplished its objectives, by answering questions that have been open for a while. The transitivity and primitivity of the actions of the alternating group A_n on $X^{[r]}$ and $X^{(r)}$, respectively the ordered and unordered r-element subsets of $X = \{1, 2, \cdots, n\}$, have been determined. Also, the ranks and subdegrees of the actions have been calculated, and the pairing of the corresponding suborbits explored. Additionally, the suborbital graphs associated with the actions have been constructed and examination of their theoretic properties done. Besides, an expression of the cycle index formula of the symmetric group S_n explicitly in terms of the cycle indices of the alternating group A_n and the cyclic group C_2 has been obtained.

8.4 Recommendations for Further Research

Having achieved the objectives of the current study, there remain other interesting areas related to the study that have not received any attention. One may investigate the transitivity, primitivity, ranks, subdegrees and suborbital graphs of any of the following actions, for instance.

1. The Cartesian product $S_n \times A_n$ of the symmetric group S_n by the alternating group A_n on $X^{(r)} \times X^{(r)}$ or $X^{[r]} \times X^{[r]}$.
2. The dihedral group D_n and the cyclic group C_n on the diagonals of a regular n-gon.
3. The group of units of $\mathbb{Z}_n$ on the set $\mathbb{Z}_n \setminus \{0\}$ of non-zero elements of $\mathbb{Z}_n$.

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APPENDICES

Appendix A: Suborbits of $\mathbf{A_n}\,(\mathbf{n} \geq 8)$ Acting on $\mathbf{X^{[3]}}$

$\triangle_0 = Orb_{G_{[1,2,3]}}[1,2,3] = \{[1,2,3]\}$
$\triangle_1 = Orb_{G_{[1,2,3]}}[1,3,2] = \{[1,3,2]\}$
$\triangle_2 = Orb_{G_{[1,2,3]}}[2,1,3] = \{[2,1,3]\}$
$\triangle_3 = Orb_{G_{[1,2,3]}}[2,3,1] = \{[2,3,1]\}$
$\triangle_4 = Orb_{G_{[1,2,3]}}[3,1,2] = \{[3,1,2]\}$
$\triangle_5 = Orb_{G_{[1,2,3]}}[3,2,1] = \{[3,2,1]\}$
$\triangle_6 = Orb_{G_{[1,2,3]}}[1,2,4] = \{[1,2,4],[1,2,5],[1,2,6],\cdots,[1,2,n]\}$
$\triangle_7 = Orb_{G_{[1,2,3]}}[1,4,2] = \{[1,4,2],[1,5,2],[1,6,2],\cdots,[1,n,2]\}$
$\triangle_8 = Orb_{G_{[1,2,3]}}[2,1,4] = \{[2,1,4],[2,1,5],[2,1,6],\cdots,[2,1,n]\}$
$\triangle_9 = Orb_{G_{[1,2,3]}}[2,4,1] = \{[2,4,1],[2,5,1],[2,6,1],\cdots,[2,n,1]\}$
$\triangle_{10} = Orb_{G_{[1,2,3]}}[4,1,2] = \{[4,1,2],[5,1,2],[6,1,2],\cdots,[n,1,2]\}$
$\triangle_{11} = Orb_{G_{[1,2,3]}}[4,2,1] = \{[4,2,1],[5,2,1],[6,2,1],\cdots,[n,2,1]\}$
$\triangle_{12} = Orb_{G_{[1,2,3]}}[1,3,4] = \{[1,3,4],[1,3,5],[1,3,6],\cdots,[1,3,n]\}$
$\triangle_{13} = Orb_{G_{[1,2,3]}}[1,4,3] = \{[1,4,3],[1,5,3],[1,6,3],\cdots,[1,n,3]\}$
$\triangle_{14} = Orb_{G_{[1,2,3]}}[3,1,4] = \{[3,1,4],[3,1,5],[3,1,6],\cdots,[3,1,n]\}$
$\triangle_{15} = Orb_{G_{[1,2,3]}}[3,4,1] = \{[3,4,1],[3,5,1],[3,6,1],\cdots,[3,n,1]\}$
$\triangle_{16} = Orb_{G_{[1,2,3]}}[4,1,3] = \{[4,1,3],[5,1,3],[6,1,3],\cdots,[n,1,3]\}$
$\triangle_{17} = Orb_{G_{[1,2,3]}}[4,3,1] = \{[4,3,1],[5,3,1],[6,3,1],\cdots,[n,3,1]\}$
$\triangle_{18} = Orb_{G_{[1,2,3]}}[2,3,4] = \{[2,3,4],[2,3,5],[2,3,6],\cdots,[2,3,n]\}$
$\triangle_{19} = Orb_{G_{[1,2,3]}}[2,4,3] = \{[2,4,3],[2,5,3],[2,6,3],\cdots,[2,n,3]\}$
$\triangle_{20} = Orb_{G_{[1,2,3]}}[3,2,4] = \{[3,2,4],[3,2,5],[3,2,6],\cdots,[3,2,n]\}$
$\triangle_{21} = Orb_{G_{[1,2,3]}}[3,4,2] = \{[3,4,2],[3,5,2],[3,6,2],\cdots,[3,n,2]\}$
$\triangle_{22} = Orb_{G_{[1,2,3]}}[4,2,3] = \{[4,2,3],[5,2,3],[6,2,3],\cdots,[n,2,3]\}$
$\triangle_{23} = Orb_{G_{[1,2,3]}}[4,3,2] = \{[4,3,2],[5,3,2],[6,3,2],\cdots,[n,3,2]\}$
$\triangle_{24} = Orb_{G_{[1,2,3]}}[1,4,5] = \{[1,4,5],[1,4,6],\cdots,[1,4,n],[1,5,4],[1,5,6],\cdots,[1,n,n-1]\}$
$\triangle_{25} = Orb_{G_{[1,2,3]}}[4,1,5] = \{[4,1,5],[4,1,6],\cdots,[4,1,n],[5,1,4],[5,1,6],\cdots,[n,1,n-1]\}$
$\triangle_{26} = Orb_{G_{[1,2,3]}}[4,5,1] = \{4,5,1],[4,6,1],\cdots,[4,n,1],[5,4,1],[5,6,1],\cdots,[n,n-1,1]\}$
$\triangle_{27} = Orb_{G_{[1,2,3]}}[2,4,5] = \{[2,4,5],[2,4,6],\cdots,[2,4,n],[2,5,4],[2,5,6],\cdots,[2,n,n-1]\}$
$\triangle_{28} = Orb_{G_{[1,2,3]}}[4,2,5] = \{[4,2,5],[4,2,6],\cdots,[4,2,n],[5,2,4],[5,2,6],\cdots,[n,2,n-1]\}$
$\triangle_{29} = Orb_{G_{[1,2,3]}}[4,5,2] = \{[4,5,2],[4,6,2],\cdots,[4,n,2],[5,4,2],[5,6,2],\cdots,[n,n-1,2]\}$
$\triangle_{30} = Orb_{G_{[1,2,3]}}[3,4,5] = \{[3,4,5],[3,4,6],\cdots,[3,4,n],[3,5,4],[3,5,6],\cdots,[3,n,n-1]\}$
$\triangle_{31} = Orb_{G_{[1,2,3]}}[4,3,5] = \{[4,3,5],[4,3,6],\cdots,[4,3,n],[5,3,4],[5,3,6],\cdots,[n,3,n-1]\}$
$\triangle_{32} = Orb_{G_{[1,2,3]}}[4,5,3] = \{[4,5,3],[4,6,3],\cdots,[4,n,3],[5,4,3],[5,6,3],\cdots,[n,n-1,3]\}$

$$
\begin{aligned}
\triangle_{33} = Orb_{G_{[1,2,3]}}[4,5,6] = \{&[4,5,6],[4,5,7],\cdots,[4,5,n],\\
&[4,6,5],[4,6,7],\cdots,[4,n,n-1],[5,4,6],\cdots,[n,n-1,n-2]\}
\end{aligned}
$$

Appendix B: Suborbits of $\mathbf{A_n}\ (\mathbf{n} \geq 10)$ Acting on $\mathbf{X^{[4]}}$

$\triangle_0 = Orb_{G_{[1,2,3,4]}}[1,2,3,4] = \{[1,2,3,4]\}$
$\triangle_1 = Orb_{G_{[1,2,3,4]}}[1,2,4,3] = \{[1,2,4,3]\}$
$\triangle_2 = Orb_{G_{[1,2,3,4]}}[1,3,2,4] = \{[1,3,2,4]\}$
$\triangle_3 = Orb_{G_{[1,2,3,4]}}[1,3,4,2] = \{[1,3,4,2]\}$
$\triangle_4 = Orb_{G_{[1,2,3,4]}}[1,4,2,3] = \{[1,4,2,3]\}$
$\triangle_5 = Orb_{G_{[1,2,3,4]}}[1,4,3,2] = \{[1,4,3,2]\}$
$\triangle_6 = Orb_{G_{[1,2,3,4]}}[2,1,3,4] = \{[2,1,3,4]\}$
$\triangle_7 = Orb_{G_{[1,2,3,4]}}[2,1,4,3] = \{[2,1,4,3]\}$
$\triangle_8 = Orb_{G_{[1,2,3,4]}}[2,3,1,4] = \{[2,3,1,4]\}$
$\triangle_9 = Orb_{G_{[1,2,3,4]}}[2,3,4,1] = \{[2,3,4,1]\}$
$\triangle_{10} = Orb_{G_{[1,2,3,4]}}[2,4,1,3] = \{[2,4,1,3]\}$
$\triangle_{11} = Orb_{G_{[1,2,3,4]}}[2,4,3,1] = \{[2,4,3,1]\}$
$\triangle_{12} = Orb_{G_{[1,2,3,4]}}[3,1,2,4] = \{[3,1,2,4]\}$
$\triangle_{13} = Orb_{G_{[1,2,3,4]}}[3,1,4,2] = \{[3,1,4,2]\}$
$\triangle_{14} = Orb_{G_{[1,2,3,4]}}[3,2,1,4] = \{[3,2,1,4]\}$
$\triangle_{15} = Orb_{G_{[1,2,3,4]}}[3,2,4,1] = \{[3,2,4,1]\}$
$\triangle_{16} = Orb_{G_{[1,2,3,4]}}[3,4,1,2] = \{[3,4,1,2]\}$
$\triangle_{17} = Orb_{G_{[1,2,3,4]}}[3,4,2,1] = \{[3,4,2,1]\}$
$\triangle_{18} = Orb_{G_{[1,2,3,4]}}[4,1,2,3] = \{[4,1,2,3]\}$
$\triangle_{19} = Orb_{G_{[1,2,3,4]}}[4,1,3,2] = \{[4,1,3,2]\}$
$\triangle_{20} = Orb_{G_{[1,2,3,4]}}[4,2,1,3] = \{[4,2,1,3]\}$
$\triangle_{21} = Orb_{G_{[1,2,3,4]}}[4,2,3,1] = \{[4,2,3,1]\}$
$\triangle_{22} = Orb_{G_{[1,2,3,4]}}[4,3,1,2] = \{[4,3,1,2]\}$
$\triangle_{23} = Orb_{G_{[1,2,3,4]}}[4,3,2,1] = \{[4,3,2,1]\}$
$\triangle_{24} = Orb_{G_{[1,2,3,4]}}[1,2,3,5] = \{[1,2,3,5],[1,2,3,6],\cdots,[1,2,3,n]\}$
$\triangle_{25} = Orb_{G_{[1,2,3,4]}}[1,2,5,3] = \{[1,2,5,3],[1,2,6,3],\cdots,[1,2,n,3]\}$
$\triangle_{26} = Orb_{G_{[1,2,3,4]}}[1,5,2,3] = \{[1,5,2,3],[1,6,2,3],\cdots,[1,n,2,3]\}$
$\triangle_{27} = Orb_{G_{[1,2,3,4]}}[1,5,3,2] = \{[1,5,3,2],[1,6,3,2],\cdots,[1,n,3,2]\}$
$\triangle_{28} = Orb_{G_{[1,2,3,4]}}[1,3,2,5] = \{[1,3,2,5],[1,3,2,6],\cdots,[1,3,2,n]\}$
$\triangle_{29} = Orb_{G_{[1,2,3,4]}}[1,3,5,2] = \{[1,3,5,2],[1,3,6,2],\cdots,[1,3,n,2]\}$
$\triangle_{30} = Orb_{G_{[1,2,3,4]}}[2,1,3,5] = \{[2,1,3,5],[2,1,3,6],\cdots,[2,1,3,n]\}$
$\triangle_{31} = Orb_{G_{[1,2,3,4]}}[2,1,5,3] = \{[2,1,5,3],[2,1,6,3],\cdots,[2,1,n,3]\}$
$\triangle_{32} = Orb_{G_{[1,2,3,4]}}[2,3,1,5] = \{[2,3,1,5],[2,3,1,6],\cdots,[2,3,1,n]\}$
$\triangle_{33} = Orb_{G_{[1,2,3,4]}}[2,3,5,1] = \{[2,3,5,1],[2,3,6,1],\cdots,[2,3,n,1]\}$

$$\triangle_{34} = Orb_{G_{[1,2,3,4]}}[2,5,1,3] = \{[2,5,1,3],[2,6,1,3],\cdots,[2,n,1,3]\}$$
$$\triangle_{35} = Orb_{G_{[1,2,3,4]}}[2,5,3,1] = \{[2,5,3,1],[2,6,3,1],\cdots,[2,n,3,1]\}$$
$$\triangle_{36} = Orb_{G_{[1,2,3,4]}}[3,1,2,5] = \{[3,1,2,5],[3,1,2,6],\cdots,[3,1,2,n]\}$$
$$\triangle_{37} = Orb_{G_{[1,2,3,4]}}[3,1,5,2] = \{[3,1,5,2],[3,1,6,2],\cdots,[3,1,n,2]\}$$
$$\triangle_{38} = Orb_{G_{[1,2,3,4]}}[3,5,1,2] = \{[3,5,1,2],[3,6,1,2],\cdots,[3,n,1,2]\}$$
$$\triangle_{39} = Orb_{G_{[1,2,3,4]}}[3,5,2,1] = \{[3,5,2,1],[3,6,2,1],\cdots,[3,n,2,1]\}$$
$$\triangle_{40} = Orb_{G_{[1,2,3,4]}}[3,2,1,5] = \{[3,2,1,5],[3,2,1,6],\cdots,[3,2,1,n]\}$$
$$\triangle_{41} = Orb_{G_{[1,2,3,4]}}[3,2,5,1] = \{[3,2,5,1],[3,2,6,1],\cdots,[3,2,n,1]\}$$
$$\triangle_{42} = Orb_{G_{[1,2,3,4]}}[5,1,2,3] = \{[5,1,2,3],[6,1,2,3],\cdots,[n,1,2,3]\}$$
$$\triangle_{43} = Orb_{G_{[1,2,3,4]}}[5,1,3,2] = \{[5,1,3,2],[6,1,3,2],\cdots,[n,1,3,2]\}$$
$$\triangle_{44} = Orb_{G_{[1,2,3,4]}}[5,2,1,3] = \{[5,2,1,3],[6,2,1,3],\cdots,[n,2,1,3]\}$$
$$\triangle_{45} = Orb_{G_{[1,2,3,4]}}[5,2,3,1] = \{[5,2,3,1],[6,2,3,1],\cdots,[n,2,3,1]\}$$
$$\triangle_{46} = Orb_{G_{[1,2,3,4]}}[5,3,1,2] = \{[5,3,1,2],[6,3,1,2],\cdots,[n,3,1,2]\}$$
$$\triangle_{47} = Orb_{G_{[1,2,3,4]}}[5,3,2,1] = \{[5,3,2,1],[6,3,2,1],\cdots,[n,3,2,1]\}$$
$$\triangle_{48} = Orb_{G_{[1,2,3,4]}}[1,2,4,5] = \{[1,2,4,5],[1,2,4,6],\cdots,[1,2,4,n]\}$$
$$\triangle_{49} = Orb_{G_{[1,2,3,4]}}[1,2,5,4] = \{[1,2,5,4],[1,2,6,4],\cdots,[1,2,n,4]\}$$
$$\triangle_{50} = Orb_{G_{[1,2,3,4]}}[1,4,2,5] = \{[1,4,2,5],[1,4,2,6],\cdots,[1,4,2,n]\}$$
$$\triangle_{51} = Orb_{G_{[1,2,3,4]}}[1,4,5,2] = \{[1,4,5,2],[1,4,6,2],\cdots,[1,4,n,2]\}$$
$$\triangle_{52} = Orb_{G_{[1,2,3,4]}}[1,5,2,4] = \{[1,5,2,4],[1,6,2,4],\cdots,[1,n,2,4]\}$$
$$\triangle_{53} = Orb_{G_{[1,2,3,4]}}[1,5,4,2] = \{[1,5,4,2],[1,6,4,2],\cdots,[1,n,4,2]\}$$
$$\triangle_{54} = Orb_{G_{[1,2,3,4]}}[2,1,4,5] = \{[2,1,4,5],[2,1,4,6],\cdots,[2,1,4,n]\}$$
$$\triangle_{55} = Orb_{G_{[1,2,3,4]}}[2,1,5,4] = \{[2,1,5,4],[2,1,6,4],\cdots,[2,1,n,4]\}$$
$$\triangle_{56} = Orb_{G_{[1,2,3,4]}}[2,4,1,5] = \{[2,4,1,5],[2,4,1,6],\cdots,[2,4,1,n]\}$$
$$\triangle_{57} = Orb_{G_{[1,2,3,4]}}[2,4,5,1] = \{[2,4,5,1],[2,4,6,1],\cdots,[2,4,n,1]\}$$
$$\triangle_{58} = Orb_{G_{[1,2,3,4]}}[2,5,1,4] = \{[2,5,1,4],[2,6,1,4],\cdots,[2,n,1,4]\}$$
$$\triangle_{59} = Orb_{G_{[1,2,3,4]}}[2,5,4,1] = \{[2,5,4,1],[2,6,4,1],\cdots,[2,n,4,1]\}$$
$$\triangle_{60} = Orb_{G_{[1,2,3,4]}}[4,1,2,5] = \{[4,1,2,5],[4,1,2,6],\cdots,[4,1,2,n]\}$$
$$\triangle_{61} = Orb_{G_{[1,2,3,4]}}[4,1,5,2] = \{[4,1,5,2],[4,1,6,2],\cdots,[4,1,n,2]\}$$
$$\triangle_{62} = Orb_{G_{[1,2,3,4]}}[4,2,1,5] = \{[4,2,1,5],[4,2,1,6],\cdots,[4,2,1,n]\}$$
$$\triangle_{63} = Orb_{G_{[1,2,3,4]}}[4,2,5,1] = \{[4,2,5,1],[4,2,6,1],\cdots,[4,2,n,1]\}$$
$$\triangle_{64} = Orb_{G_{[1,2,3,4]}}[4,5,1,2] = \{[4,5,1,2],[4,6,1,2],\cdots,[4,n,1,2]\}$$
$$\triangle_{65} = Orb_{G_{[1,2,3,4]}}[4,5,2,1] = \{[4,5,2,1],[4,6,2,1],\cdots,[4,n,2,1]\}$$
$$\triangle_{66} = Orb_{G_{[1,2,3,4]}}[5,1,2,4] = \{[5,1,2,4],[6,1,2,4],\cdots,[n,1,2,4]\}$$
$$\triangle_{67} = Orb_{G_{[1,2,3,4]}}[5,1,4,2] = \{[5,1,4,2],[6,1,4,2],\cdots,[n,1,4,2]\}$$
$$\triangle_{68} = Orb_{G_{[1,2,3,4]}}[5,2,1,4] = \{[5,2,1,4],[6,2,1,4],\cdots,[n,2,1,4]\}$$
$$\triangle_{69} = Orb_{G_{[1,2,3,4]}}[5,2,4,1] = \{[5,2,4,1],[6,2,4,1],\cdots,[n,2,4,1]\}$$
$$\triangle_{70} = Orb_{G_{[1,2,3,4]}}[5,4,1,2] = \{[5,4,1,2],[6,4,1,2],\cdots,[n,4,1,2]\}$$
$$\triangle_{71} = Orb_{G_{[1,2,3,4]}}[5,4,2,1] = \{[5,4,2,1],[6,4,2,1],\cdots,[n,4,2,1]\}$$

$$\triangle_{72} = Orb_{G_{[1,2,3,4]}}[1,3,4,5] = \{[1,3,4,5],[1,3,4,6],\cdots,[1,3,4,n]\}$$
$$\triangle_{73} = Orb_{G_{[1,2,3,4]}}[1,3,5,4] = \{[1,3,5,4],[1,3,6,4],\cdots,[1,3,n,4]\}$$
$$\triangle_{74} = Orb_{G_{[1,2,3,4]}}[1,4,3,5] = \{[1,4,3,5],[1,4,3,6],\cdots,[1,4,3,n]\}$$
$$\triangle_{75} = Orb_{G_{[1,2,3,4]}}[1,4,5,3] = \{[1,4,5,3],[1,4,6,3],\cdots,[1,4,n,3]\}$$
$$\triangle_{76} = Orb_{G_{[1,2,3,4]}}[1,5,3,4] = \{[1,5,3,4],[1,6,3,4],\cdots,[1,n,3,4]\}$$
$$\triangle_{77} = Orb_{G_{[1,2,3,4]}}[1,5,4,3] = \{[1,5,4,3],[1,6,4,3],\cdots,[1,n,4,3]\}$$
$$\triangle_{78} = Orb_{G_{[1,2,3,4]}}[3,1,4,5] = \{[3,1,4,5],[3,1,4,6],\cdots,[3,1,4,n]\}$$
$$\triangle_{79} = Orb_{G_{[1,2,3,4]}}[3,1,5,4] = \{[3,1,5,4],[3,1,6,4],\cdots,[3,1,n,4]\}$$
$$\triangle_{80} = Orb_{G_{[1,2,3,4]}}[3,4,1,5] = \{[3,4,1,5],[3,4,1,6],\cdots,[3,4,1,n]\}$$
$$\triangle_{81} = Orb_{G_{[1,2,3,4]}}[3,4,5,1] = \{[3,4,5,1],[3,4,6,1],\cdots,[3,4,n,1]\}$$
$$\triangle_{82} = Orb_{G_{[1,2,3,4]}}[3,5,1,4] = \{[3,5,1,4],[3,6,1,4],\cdots,[3,n,1,4]\}$$
$$\triangle_{83} = Orb_{G_{[1,2,3,4]}}[3,5,4,1] = \{[3,5,4,1],[3,6,4,1],\cdots,[3,n,4,1]\}$$
$$\triangle_{84} = Orb_{G_{[1,2,3,4]}}[4,1,3,5] = \{[4,1,3,5],[4,1,3,6],\cdots,[4,1,3,n]\}$$
$$\triangle_{85} = Orb_{G_{[1,2,3,4]}}[4,1,5,3] = \{[4,1,5,3],[4,1,6,3],\cdots,[4,1,n,3]\}$$
$$\triangle_{86} = Orb_{G_{[1,2,3,4]}}[4,3,1,5] = \{[4,3,1,5],[4,3,1,6],\cdots,[4,3,1,n]\}$$
$$\triangle_{87} = Orb_{G_{[1,2,3,4]}}[4,3,5,1] = \{[4,3,5,1],[4,3,6,1],\cdots,[4,3,n,1]\}$$
$$\triangle_{88} = Orb_{G_{[1,2,3,4]}}[4,5,1,3] = \{[4,5,1,3],[4,6,1,3],\cdots,[4,n,1,3]\}$$
$$\triangle_{89} = Orb_{G_{[1,2,3,4]}}[4,5,3,1] = \{[4,5,3,1],[4,6,3,1],\cdots,[4,n,3,1]\}$$
$$\triangle_{90} = Orb_{G_{[1,2,3,4]}}[5,1,3,4] = \{[5,1,3,4],[6,1,3,4],\cdots,[n,1,3,4]\}$$
$$\triangle_{91} = Orb_{G_{[1,2,3,4]}}[5,1,4,3] = \{[5,1,4,3],[6,1,4,3],\cdots,[n,1,4,3]\}$$
$$\triangle_{92} = Orb_{G_{[1,2,3,4]}}[5,3,1,4] = \{[5,3,1,4],[6,3,1,4],\cdots,[n,3,1,4]\}$$
$$\triangle_{93} = Orb_{G_{[1,2,3,4]}}[5,3,4,1] = \{[5,3,4,1],[6,3,4,1],\cdots,[n,3,4,1]\}$$
$$\triangle_{94} = Orb_{G_{[1,2,3,4]}}[5,4,1,3] = \{[5,4,1,3],[6,4,1,3],\cdots,[n,4,1,3]\}$$
$$\triangle_{95} = Orb_{G_{[1,2,3,4]}}[5,4,3,1] = \{[5,4,3,1],[6,4,3,1],\cdots,[n,4,3,1]\}$$
$$\triangle_{96} = Orb_{G_{[1,2,3,4]}}[2,3,4,5] = \{[2,3,4,5],[2,3,4,6],\cdots,[2,3,4,n]\}$$
$$\triangle_{97} = Orb_{G_{[1,2,3,4]}}[2,3,5,4] = \{[2,3,5,4],[2,3,6,4],\cdots,[2,3,n,4]\}$$
$$\triangle_{98} = Orb_{G_{[1,2,3,4]}}[2,4,3,5] = \{[2,4,3,5],[2,4,3,6],\cdots,[2,4,3,n]\}$$
$$\triangle_{99} = Orb_{G_{[1,2,3,4]}}[2,4,5,3] = \{[2,4,5,3],[2,4,6,3],\cdots,[2,4,n,3]\}$$
$$\triangle_{100} = Orb_{G_{[1,2,3,4]}}[2,5,3,4] = \{[2,5,3,4],[2,6,3,4],\cdots,[2,n,3,4]\}$$
$$\triangle_{101} = Orb_{G_{[1,2,3,4]}}[2,5,4,3] = \{[2,5,4,3],[2,6,4,3],\cdots,[2,n,4,3]\}$$
$$\triangle_{102} = Orb_{G_{[1,2,3,4]}}[3,2,4,5] = \{[3,2,4,5],[3,2,4,6],\cdots,[3,2,4,n]\}$$
$$\triangle_{103} = Orb_{G_{[1,2,3,4]}}[3,2,5,4] = \{[3,2,5,4],[3,2,6,4],\cdots,[3,2,n,4]\}$$
$$\triangle_{104} = Orb_{G_{[1,2,3,4]}}[3,4,2,5] = \{[3,4,2,5],[3,4,2,6],\cdots,[3,4,2,n]\}$$
$$\triangle_{105} = Orb_{G_{[1,2,3,4]}}[3,4,5,2] = \{[3,4,5,2],[3,4,6,2],\cdots,[3,4,n,2]\}$$
$$\triangle_{106} = Orb_{G_{[1,2,3,4]}}[3,5,2,4] = \{[3,5,2,4],[3,6,2,4],\cdots,[3,n,2,4]\}$$
$$\triangle_{107} = Orb_{G_{[1,2,3,4]}}[3,5,4,2] = \{[3,5,4,2],[3,6,4,2],\cdots,[3,n,4,2]\}$$
$$\triangle_{108} = Orb_{G_{[1,2,3,4]}}[4,2,3,5] = \{[4,2,3,5],[4,2,3,6],\cdots,[4,2,3,n]\}$$
$$\triangle_{109} = Orb_{G_{[1,2,3,4]}}[4,2,5,3] = \{[4,2,5,3],[4,2,6,3],\cdots,[4,2,n,3]\}$$

$$\triangle_{110} = Orb_{G_{[1,2,3,4]}}[4,3,2,5] = \{[4,3,2,5],[4,3,2,6],\cdots,[4,3,2,n]\}$$
$$\triangle_{111} = Orb_{G_{[1,2,3,4]}}[4,3,5,2] = \{[4,3,5,2],[4,3,6,2],\cdots,[4,3,n,2]\}$$
$$\triangle_{112} = Orb_{G_{[1,2,3,4]}}[4,5,2,3] = \{[4,5,2,3],[4,6,2,3],\cdots,[4,n,2,3]\}$$
$$\triangle_{113} = Orb_{G_{[1,2,3,4]}}[4,5,3,2] = \{[4,5,3,2],[4,6,3,2],\cdots,[4,n,3,2]\}$$
$$\triangle_{114} = Orb_{G_{[1,2,3,4]}}[5,2,3,4] = \{[5,2,3,4],[6,2,3,4],\cdots,[n,2,3,4]\}$$
$$\triangle_{115} = Orb_{G_{[1,2,3,4]}}[5,2,4,3] = \{[5,2,4,3],[6,2,4,3],\cdots,[n,2,4,3]\}$$
$$\triangle_{116} = Orb_{G_{[1,2,3,4]}}[5,3,2,4] = \{[5,3,2,4],[6,3,2,4],\cdots,[n,3,2,4]\}$$
$$\triangle_{117} = Orb_{G_{[1,2,3,4]}}[5,3,4,2] = \{[5,3,4,2],[6,3,4,2],\cdots,[n,3,4,2]\}$$
$$\triangle_{118} = Orb_{G_{[1,2,3,4]}}[5,4,2,3] = \{[5,4,2,3],[6,4,2,3],\cdots,[n,4,2,3]\}$$
$$\triangle_{119} = Orb_{G_{[1,2,3,4]}}[5,4,3,2] = \{[5,4,3,2],[6,4,3,2],\cdots,[n,4,3,2]\}$$
$$\triangle_{120} = Orb_{G_{[1,2,3,4]}}[1,2,5,6] = \{[1,2,5,6],$$
$$[1,2,5,7],\cdots,[1,2,5,n],[1,2,6,5],\cdots,[1,2,n,n-1]\}$$
$$\triangle_{121} = Orb_{G_{[1,2,3,4]}}[2,1,5,6] = \{[2,1,5,6],$$
$$[2,1,5,7],\cdots,[2,1,5,n],[2,1,6,5],\cdots,[2,1,n,n-1]\}$$
$$\triangle_{122} = Orb_{G_{[1,2,3,4]}}[1,5,2,6] = \{[1,5,2,6],$$
$$[1,5,2,7],\cdots,[1,5,2,n],[1,6,2,5],\cdots,[1,n,2,n-1]\}$$
$$\triangle_{123} = Orb_{G_{[1,2,3,4]}}[2,5,1,6] = \{[2,5,1,6],$$
$$[2,5,1,7],\cdots,[2,5,1,n],[2,6,1,5],\cdots,[2,n,1,n-1]\}$$
$$\triangle_{124} = Orb_{G_{[1,2,3,4]}}[1,5,6,2] = \{[1,5,6,2],$$
$$[1,5,7,2],\cdots,[1,5,n,2],[1,6,5,2],\cdots,[1,n,n-1,2]\}$$
$$\triangle_{125} = Orb_{G_{[1,2,3,4]}}[2,5,6,1] = \{[2,5,6,1],$$
$$[2,5,7,1],\cdots,[2,5,n,1],[2,6,5,1],\cdots,[2,n,n-1,1]\}$$
$$\triangle_{126} = Orb_{G_{[1,2,3,4]}}[5,6,1,2] = \{[5,6,1,2],$$
$$[5,7,1,2],\cdots,[5,n,1,2],[6,5,1,2],\cdots,[n,n-1,1,2]\}$$
$$\triangle_{127} = Orb_{G_{[1,2,3,4]}}[5,6,2,1] = \{[5,6,2,1],$$
$$[5,7,2,1],\cdots,[5,n,2,1],[6,5,2,1],\cdots,[n,n-1,2,1]\}$$
$$\triangle_{128} = Orb_{G_{[1,2,3,4]}}[5,1,6,2] = \{[5,1,6,2],$$
$$[5,1,7,2],\cdots,[5,1,n,2],[6,1,5,2],\cdots,[n,1,n-1,2]\}$$
$$\triangle_{129} = Orb_{G_{[1,2,3,4]}}[5,2,6,1] = \{[5,2,6,1],$$
$$[5,2,7,1],\cdots,[5,2,n,1],[6,2,5,1],\cdots,[n,2,n-1,1]\}$$
$$\triangle_{130} = Orb_{G_{[1,2,3,4]}}[5,1,2,6] = \{[5,1,2,6],$$
$$[5,1,2,7],\cdots,[5,1,2,n],[6,1,2,5],\cdots,[n,1,2,n-1]\}$$
$$\triangle_{131} = Orb_{G_{[1,2,3,4]}}[5,2,1,6] = \{[5,2,1,6],$$
$$[5,2,1,7],\cdots,[5,2,1,n],[6,2,1,5],\cdots,[n,2,1,n-1]\}$$
$$\triangle_{132} = Orb_{G_{[1,2,3,4]}}[1,3,5,6] = \{[1,3,5,6],$$
$$[1,3,5,7],\cdots,[1,3,5,n],[1,3,6,5],\cdots,[1,3,n,n-1]\}$$
$$\triangle_{133} = Orb_{G_{[1,2,3,4]}}[3,1,5,6] = \{[3,1,5,6],$$
$$[3,1,5,7],\cdots,[3,1,5,n],[3,1,6,5],\cdots,[3,1,n,n-1]\}$$

$$\triangle_{134} = Orb_{G_{[1,2,3,4]}}[1,5,3,6] = \{[1,5,3,6],$$
$$[1,5,3,7],\cdots,[1,5,3,n],[1,6,3,5],\cdots,[1,n,3,n-1]\}$$
$$\triangle_{135} = Orb_{G_{[1,2,3,4]}}[3,5,1,6] = \{[3,5,1,6],$$
$$[3,5,1,7],\cdots,[3,5,1,n],[3,6,1,5],\cdots,[3,n,1,n-1]\}$$
$$\triangle_{136} = Orb_{G_{[1,2,3,4]}}[1,5,6,3] = \{[1,5,6,3],$$
$$[1,5,7,3],\cdots,[1,5,n,3],[1,6,5,3],\cdots,[1,n,n-1,3]\}$$
$$\triangle_{137} = Orb_{G_{[1,2,3,4]}}[3,5,6,1] = \{[3,5,6,1],$$
$$[3,5,7,1],\cdots,[3,5,n,1],[3,6,5,1],\cdots,[3,n,n-1,1]\}$$
$$\triangle_{138} = Orb_{G_{[1,2,3,4]}}[5,6,1,3] = \{[5,6,1,3],$$
$$[5,7,1,3],\cdots,[5,n,1,3],[6,5,1,3],\cdots,[n,n-1,1,3]\}$$
$$\triangle_{139} = Orb_{G_{[1,2,3,4]}}[5,6,3,1] = \{[5,6,3,1],$$
$$[5,7,3,1],\cdots,[5,n,3,1],[6,5,3,1],\cdots,[n,n-1,3,1]\}$$
$$\triangle_{140} = Orb_{G_{[1,2,3,4]}}[5,1,6,3] = \{[5,1,6,3],$$
$$[5,1,7,3],\cdots,[5,1,n,3],[6,1,5,3],\cdots,[n,1,n-1,3]\}$$
$$\triangle_{141} = Orb_{G_{[1,2,3,4]}}[5,3,6,1] = \{[5,3,6,1],$$
$$[5,3,7,1],\cdots,[5,3,n,1],[6,3,5,1],\cdots,[n,3,n-1,1]\}$$
$$\triangle_{142} = Orb_{G_{[1,2,3,4]}}[5,1,3,6] = \{[5,1,3,6],$$
$$[5,1,3,7],\cdots,[5,1,3,n],[6,1,3,5],\cdots,[n,1,3,n-1]\}$$
$$\triangle_{143} = Orb_{G_{[1,2,3,4]}}[5,3,1,6] = \{[5,3,1,6],$$
$$[5,3,1,7],\cdots,[5,3,1,n],[6,3,1,5],\cdots,[n,3,1,n-1]\}$$
$$\triangle_{144} = Orb_{G_{[1,2,3,4]}}[1,4,5,6] = \{[1,4,5,6],$$
$$[1,4,5,7],\cdots,[1,4,5,n],[1,4,6,5],\cdots,[1,4,n,n-1]\}$$
$$\triangle_{145} = Orb_{G_{[1,2,3,4]}}[4,1,5,6] = \{[4,1,5,6],$$
$$[4,1,5,7],\cdots,[4,1,5,n],[4,1,6,5],\cdots,[4,1,n,n-1]\}$$
$$\triangle_{146} = Orb_{G_{[1,2,3,4]}}[1,5,4,6] = \{[1,5,4,6],$$
$$[1,5,4,7],\cdots,[1,5,4,n],[1,6,4,5],\cdots,[1,n,4,n-1]\}$$
$$\triangle_{147} = Orb_{G_{[1,2,3,4]}}[4,5,1,6] = \{[4,5,1,6],$$
$$[4,5,1,7],\cdots,[4,5,1,n],[4,6,1,5],\cdots,[4,n,1,n-1]\}$$
$$\triangle_{148} = Orb_{G_{[1,2,3,4]}}[1,5,6,4] = \{[1,5,6,4],$$
$$[1,5,7,4],\cdots,[1,5,n,4],[1,6,5,4],\cdots,[1,n,n-1,4]\}$$
$$\triangle_{149} = Orb_{G_{[1,2,3,4]}}[4,5,6,1] = \{[4,5,6,1],$$
$$[4,5,7,1],\cdots,[4,5,n,1],[4,6,5,1],\cdots,[4,n,n-1,1]\}$$
$$\triangle_{150} = Orb_{G_{[1,2,3,4]}}[5,1,4,6] = \{[5,1,4,6],$$
$$[5,1,4,7],\cdots,[5,1,4,n],[6,1,4,5],\cdots,[n,1,4,n-1]\}$$
$$\triangle_{151} = Orb_{G_{[1,2,3,4]}}[5,4,1,6] = \{[5,4,1,6],$$
$$[5,4,1,7],\cdots,[5,4,1,n],[6,4,1,5],\cdots,[n,4,1,n-1]\}$$
$$\triangle_{152} = Orb_{G_{[1,2,3,4]}}[5,1,6,4] = \{[5,1,6,4],$$
$$[5,1,7,4],\cdots,[5,1,n,4],[6,1,5,4],\cdots,[n,1,n-1,4]\}$$

$$\triangle_{153} = Orb_{G_{[1,2,3,4]}}[5,4,6,1] = \{[5,4,6,1],$$
$$[5,4,7,1],\cdots,[5,4,n,1],[6,4,5,1],\cdots,[n,4,n-1,1]\}$$
$$\triangle_{154} = Orb_{G_{[1,2,3,4]}}[5,6,1,4] = \{[5,6,1,4],$$
$$[5,7,1,4],\cdots,[5,n,1,4],[6,5,1,4],\cdots,[n,n-1,1,4]\}$$
$$\triangle_{155} = Orb_{G_{[1,2,3,4]}}[5,6,4,1] = \{[5,6,4,1],$$
$$[5,7,4,1],\cdots,[5,n,4,1],[6,5,4,1],\cdots,[n,n-1,4,1]\}$$
$$\triangle_{156} = Orb_{G_{[1,2,3,4]}}[2,3,5,6] = \{[2,3,5,6],$$
$$[2,3,5,7],\cdots,[2,3,5,n],[2,3,6,5],\cdots,[2,3,n,n-1]\}$$
$$\triangle_{157} = Orb_{G_{[1,2,3,4]}}[3,2,5,6] = \{[3,2,5,6],$$
$$[3,2,5,7],\cdots,[3,2,5,n],[3,2,6,5],\cdots,[3,2,n,n-1]\}$$
$$\triangle_{158} = Orb_{G_{[1,2,3,4]}}[2,5,3,6] = \{[2,5,3,6],$$
$$[2,5,3,7],\cdots,[2,5,3,n],[2,6,3,5],\cdots,[2,n,3,n-1]\}$$
$$\triangle_{159} = Orb_{G_{[1,2,3,4]}}[3,5,2,6] = \{[3,5,2,6],$$
$$[3,5,2,7],\cdots,[3,5,2,n],[3,6,2,5],\cdots,[3,n,2,n-1]\}$$
$$\triangle_{160} = Orb_{G_{[1,2,3,4]}}[5,2,3,6] = \{[5,2,3,6],$$
$$[5,2,3,7],\cdots,[5,2,3,n],[6,2,3,5],\cdots,[n,2,3,n-1]\}$$
$$\triangle_{161} = Orb_{G_{[1,2,3,4]}}[5,3,2,6] = \{[5,3,2,6],$$
$$[5,3,2,7],\cdots,[5,3,2,n],[6,3,2,5],\cdots,[n,3,2,n-1]\}$$
$$\triangle_{162} = Orb_{G_{[1,2,3,4]}}[2,5,6,3] = \{[2,5,6,3],$$
$$[2,5,7,3],\cdots,[2,5,n,3],[2,6,5,3],\cdots,[2,n,n-1,3]\}$$
$$\triangle_{163} = Orb_{G_{[1,2,3,4]}}[3,5,6,2] = \{[3,5,6,2],$$
$$[3,5,7,2],\cdots,[3,5,n,2],[3,6,5,2],\cdots,[3,n,n-1,2]\}$$
$$\triangle_{164} = Orb_{G_{[1,2,3,4]}}[5,2,6,3] = \{[5,2,6,3],$$
$$[5,2,7,3],\cdots,[5,2,n,3],[6,2,5,3],\cdots,[n,2,n-1,3]\}$$
$$\triangle_{165} = Orb_{G_{[1,2,3,4]}}[5,3,6,2] = \{[5,3,6,2],$$
$$[5,3,7,2],\cdots,[5,3,n,2],[6,3,5,2],\cdots,[n,3,n-1,2]\}$$
$$\triangle_{166} = Orb_{G_{[1,2,3,4]}}[5,6,2,3] = \{[5,6,2,3],$$
$$[5,7,2,3],\cdots,[5,n,2,3],[6,5,2,3],\cdots,[n,n-1,2,3]\}$$
$$\triangle_{167} = Orb_{G_{[1,2,3,4]}}[5,6,3,2] = \{[5,6,3,2],$$
$$[5,7,3,2],\cdots,[5,n,3,2],[6,5,3,2],\cdots,[n,n-1,3,2]\}$$
$$\triangle_{168} = Orb_{G_{[1,2,3,4]}}[2,4,5,6] = \{[2,4,5,6],$$
$$[2,4,5,7],\cdots,[2,4,5,n],[2,4,6,5],\cdots,[2,4,n,n-1]\}$$
$$\triangle_{169} = Orb_{G_{[1,2,3,4]}}[4,2,5,6] = \{[4,2,5,6],$$
$$[4,2,5,7],\cdots,[4,2,5,n],[4,2,6,5],\cdots,[4,2,n,n-1]\}$$
$$\triangle_{170} = Orb_{G_{[1,2,3,4]}}[2,5,4,6] = \{[2,5,4,6],$$
$$[2,5,4,7],\cdots,[2,5,4,n],[2,6,4,5],\cdots,[2,n,4,n-1]\}$$
$$\triangle_{171} = Orb_{G_{[1,2,3,4]}}[4,5,2,6] = \{[4,5,2,6],$$
$$[4,5,2,7],\cdots,[4,5,2,n],[4,6,2,5],\cdots,[4,n,2,n-1]\}$$

$$\triangle_{172} = Orb_{G_{[1,2,3,4]}}[2,5,6,4] = \{[2,5,6,4], [2,5,7,4], \cdots, [2,5,n,4], [2,6,5,4], \cdots, [2,n,n-1,4]\}$$
$$\triangle_{173} = Orb_{G_{[1,2,3,4]}}[4,5,6,2] = \{[4,5,6,2], [4,5,7,2], \cdots, [4,5,n,2], [4,6,5,2], \cdots, [4,n,n-1,2]\}$$
$$\triangle_{174} = Orb_{G_{[1,2,3,4]}}[5,2,4,6] = \{[5,2,4,6], [5,2,4,7], \cdots, [5,2,4,n], [6,2,4,5], \cdots, [n,2,4,n-1]\}$$
$$\triangle_{175} = Orb_{G_{[1,2,3,4]}}[5,4,2,6] = \{[5,4,2,6], [5,4,2,7], \cdots, [5,4,2,n], [6,4,2,5], \cdots, [n,4,2,n-1]\}$$
$$\triangle_{176} = Orb_{G_{[1,2,3,4]}}[5,2,6,4] = \{[5,2,6,4], [5,2,7,4], \cdots, [5,2,n,4], [6,2,5,4], \cdots, [n,2,n-1,4]\}$$
$$\triangle_{177} = Orb_{G_{[1,2,3,4]}}[5,4,6,2] = \{[5,4,6,2], [5,4,7,2], \cdots, [5,4,n,2], [6,4,5,2], \cdots, [n,4,n-1,2]\}$$
$$\triangle_{178} = Orb_{G_{[1,2,3,4]}}[5,6,2,4] = \{[5,6,2,4], [5,7,2,4], \cdots, [5,n,2,4], [6,5,2,4], \cdots, [n,n-1,2,4]\}$$
$$\triangle_{179} = Orb_{G_{[1,2,3,4]}}[5,6,4,2] = \{[5,6,4,2], [5,7,4,2], \cdots, [5,n,4,2], [6,5,4,2], \cdots, [n,n-1,4,2]\}$$
$$\triangle_{180} = Orb_{G_{[1,2,3,4]}}[3,4,5,6] = \{[3,4,5,6], [3,4,5,7], \cdots, [3,4,5,n], [3,4,6,5], \cdots, [3,4,n,n-1]\}$$
$$\triangle_{181} = Orb_{G_{[1,2,3,4]}}[4,3,5,6] = \{[4,3,5,6], [4,3,5,7], \cdots, [4,3,5,n], [4,3,6,5], \cdots, [4,3,n,n-1]\}$$
$$\triangle_{182} = Orb_{G_{[1,2,3,4]}}[3,5,4,6] = \{[3,5,4,6], [3,5,4,7], \cdots, [3,5,4,n], [3,6,4,5], \cdots, [3,n,4,n-1]\}$$
$$\triangle_{183} = Orb_{G_{[1,2,3,4]}}[4,5,3,6] = \{[4,5,3,6], [4,5,3,7], \cdots, [4,5,3,n], [4,6,3,5], \cdots, [4,n,3,n-1]\}$$
$$\triangle_{184} = Orb_{G_{[1,2,3,4]}}[3,5,6,4] = \{[3,5,6,4], [3,5,7,4], \cdots, [3,5,n,4], [3,6,5,4], \cdots, [3,n,n-1,4]\}$$
$$\triangle_{185} = Orb_{G_{[1,2,3,4]}}[4,5,6,3] = \{[4,5,6,3], [4,5,7,3], \cdots, [4,5,n,3], [4,6,5,3], \cdots, [4,n,n-1,3]\}$$
$$\triangle_{186} = Orb_{G_{[1,2,3,4]}}[5,3,4,6] = \{[5,3,4,6], [5,3,4,7], \cdots, [5,3,4,n], [6,3,4,5], \cdots, [n,3,4,n-1]\}$$
$$\triangle_{187} = Orb_{G_{[1,2,3,4]}}[5,4,3,6] = \{[5,4,3,6], [5,4,3,7], \cdots, [5,4,3,n], [6,4,3,5], \cdots, [n,4,3,n-1]\}$$
$$\triangle_{188} = Orb_{G_{[1,2,3,4]}}[5,3,6,4] = \{[5,3,6,4], [5,3,7,4], \cdots, [5,3,n,4], [6,3,5,4], \cdots, [n,3,n-1,4]\}$$
$$\triangle_{189} = Orb_{G_{[1,2,3,4]}}[5,4,6,3] = \{[5,4,6,3], [5,4,7,3], \cdots, [5,4,n,3], [6,4,5,3], \cdots, [n,4,n-1,3]\}$$
$$\triangle_{190} = Orb_{G_{[1,2,3,4]}}[5,6,3,4] = \{[5,6,3,4], [5,7,3,4], \cdots, [5,n,3,4], [6,5,3,4], \cdots, [n,n-1,3,4]\}$$

$$\triangle_{191} = Orb_{G_{[1,2,3,4]}}[5,6,4,3] = \{[5,6,4,3],$$
$$[5,7,4,3], \cdots, [5,n,4,3], [6,5,4,3], \cdots, [n,n-1,4,3]\}$$
$$\triangle_{192} = Orb_{G_{[1,2,3,4]}}[1,5,6,7] = \{[1,5,6,7], [1,5,6,8], \cdots, [1,5,6,n],$$
$$[1,5,7,6], \cdots, [1,5,n,n-1], [1,6,5,7], \cdots, [1,n,n-1,n-2]\}$$
$$\triangle_{193} = Orb_{G_{[1,2,3,4]}}[5,1,6,7] = \{[5,1,6,7], [5,1,6,8], \cdots, [5,1,6,n],$$
$$[5,1,7,6], \cdots, [5,1,n,n-1], [6,1,5,7], \cdots, [n,1,n-1,n-2]\}$$
$$\triangle_{194} = Orb_{G_{[1,2,3,4]}}[5,6,1,7] = \{[5,6,1,7], [5,6,1,8], \cdots, [5,6,1,n],$$
$$[5,7,1,6], \cdots, [5,n,1,n-1], [6,5,1,7], \cdots, [n,n-1,1,n-2]\}$$
$$\triangle_{195} = Orb_{G_{[1,2,3,4]}}[5,6,7,1] = \{[5,6,7,1], [5,6,8,1], \cdots, [5,6,n,1],$$
$$[5,7,6,1], \cdots, [5,n,n-1,1], [6,5,7,1], \cdots, [n,n-1,n-2,1]\}$$
$$\triangle_{196} = Orb_{G_{[1,2,3,4]}}[2,5,6,7] = \{[2,5,6,7], [2,5,6,8], \cdots, [2,5,6,n],$$
$$[2,5,7,6], \cdots, [2,5,n,n-1], [2,6,5,7], \cdots, [2,n,n-1,n-2]\}$$
$$\triangle_{197} = Orb_{G_{[1,2,3,4]}}[5,2,6,7] = \{[5,2,6,7], [5,2,6,8], \cdots, [5,2,6,n],$$
$$[5,2,7,6], \cdots, [5,2,n,n-1], [6,2,5,7], \cdots, [n,2,n-1,n-2]\}$$
$$\triangle_{198} = Orb_{G_{[1,2,3,4]}}[5,6,2,7] = \{[5,6,2,7], [5,6,2,8], \cdots, [5,6,2,n],$$
$$[5,7,2,6], \cdots, [5,n,2,n-1], [6,5,2,7], \cdots, [n,n-1,2,n-2]\}$$
$$\triangle_{199} = Orb_{G_{[1,2,3,4]}}[5,6,7,2] = \{[5,6,7,2], [5,6,8,2], \cdots, [5,6,n,2],$$
$$[5,7,6,2], \cdots, [5,n,n-1,2], [6,5,7,2], \cdots, [n,n-1,n-2,2]\}$$
$$\triangle_{200} = Orb_{G_{[1,2,3,4]}}[3,5,6,7] = \{[3,5,6,7], [3,5,6,8], \cdots, [3,5,6,n],$$
$$[3,5,7,6], \cdots, [3,5,n,n-1], [3,6,5,7], \cdots, [3,n,n-1,n-2]\}$$
$$\triangle_{201} = Orb_{G_{[1,2,3,4]}}[5,3,6,7] = \{[5,3,6,7], [5,3,6,8], \cdots, [5,3,6,n],$$
$$[5,3,7,6], \cdots, [5,3,n,n-1], [6,3,5,7], \cdots, [n,3,n-1,n-2]\}$$
$$\triangle_{202} = Orb_{G_{[1,2,3,4]}}[5,6,3,7] = \{[5,6,3,7], [5,6,3,8], \cdots, [5,6,3,n],$$
$$[5,7,3,6], \cdots, [5,n,3,n-1], [6,5,3,7], \cdots, [n,n-1,3,n-2]\}$$
$$\triangle_{203} = Orb_{G_{[1,2,3,4]}}[5,6,7,3] = \{[5,6,7,3], [5,6,8,3], \cdots, [5,6,n,3],$$
$$[5,7,6,3], \cdots, [5,n,n-1,3], [6,5,7,3], \cdots, [n,n-1,n-2,3]\}$$
$$\triangle_{204} = Orb_{G_{[1,2,3,4]}}[4,5,6,7] = \{[4,5,6,7], [4,5,6,8], \cdots, [4,5,6,n],$$
$$[4,5,7,6], \cdots, [4,5,n,n-1], [4,6,5,7], \cdots, [4,n,n-1,n-2]\}$$
$$\triangle_{205} = Orb_{G_{[1,2,3,4]}}[5,4,6,7] = \{[5,4,6,7], [5,4,6,8], \cdots, [5,4,6,n],$$
$$[5,4,7,6], \cdots, [5,4,n,n-1], [6,4,5,7], \cdots, [n,4,n-1,n-2]\}$$
$$\triangle_{206} = Orb_{G_{[1,2,3,4]}}[5,6,4,7] = \{[5,6,4,7], [5,6,4,8], \cdots, [5,6,4,n],$$
$$[5,7,4,6], \cdots, [5,n,4,n-1], [6,5,4,7], \cdots, [n,n-1,4,n-2]\}$$
$$\triangle_{207} = Orb_{G_{[1,2,3,4]}}[5,6,7,4] = \{[5,6,7,4], [5,6,8,4], \cdots, [5,6,n,4],$$
$$[5,7,6,4], \cdots, [5,n,n-1,4], [6,5,7,4], \cdots, [n,n-1,n-2,4]\}$$
$$\triangle_{208} = Orb_{G_{[1,2,3,4]}}[5,6,7,8] = \{[5,6,7,8], \cdots, [5,6,7,n], [5,6,8,7], \cdots, [5,6,n,n-1],$$
$$[5,7,6,8], \cdots, [5,n,n-1,n-2], \cdots, [n,n-1,n-2,n-3]\}$$

Appendix C: Computer Code

```
import math
r = int(input("value of r:"))
srank = 0
i = 0
print("Number of suborbits containing exactly x elements from the set N={1,2,...,r}: ")
while i <= r:
    comb = math.factorial(r)/(math.factorial((r-(r-i)))*math.factorial(r-i))
    perm = math.factorial(r)/math.factorial(r-(r-i))
    product = comb * perm
    i += 1
    srank += product
    k = srank
    print(int(product))
print("The rank is:", int(k))
```

Appendix D: List of Publications

1. Gachimu, R., Kamuti, I., Nyaga, L. and Rimberia, J. (2015). On the Suborbits of the Alternating Group A_n Acting on Ordered r-Element Subsets. *International Electronic Journal of Pure and Applied Mathematics*, *9*(3), 137-147.

2. Gachimu, R., Kamuti, I., Nyaga, L., Rimberia, J. and Kamaku P. (2016). Properties and Invariants Associated with the Action of the Alternating Group on Unordered Subsets. *International Journal of Pure and Applied Mathematics*, *106*(1), 333-346.