Robust Optimal Portfolio and Bank Capital Adequacy Management

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MF300-0001/17

A Thesis submitted to Pan African University

Institute of Science, Technology and innovation in partial

fulfillment of the requirement for the award of the degree of Master

of Science in Mathematics (Financial option) of the Pan African University.

2018

Declaration

This thesis is my original work and has not been presented for a degree in any other University.

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Acknowledgement

Glory to Almighty God who gave me the strength and the good health until the completion of this work.

I would like to express my sincere thanks to the supervisors of this work for their guidance, encouragement, availability and knowledge direction.

I would like to express my sincere gratitude for the assistance of all the staff and the entire students of Pan African University.

Special gratitude to my parents and my family for their efforts and the sacrifices they always made since the time I reached the school bench, their love, their support and their pieces of advice.

To everyone who in his own way embellished my path of study, I am sincerely grateful.

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List of Abbreviations and Acronyms

AAPM   Ambiguity Averse Portfolio Manager

CAR   Capital Adequacy Ratio

CIR   Cox-Ingersoll and Ross

HJBI   Hamilton-Jacobi-Bellman-Isaac

RBC   Regulatory Bank Capital

RWA   Risk Weighted Asset

Abstract

The joint task of optimal portfolio selection and bank capital adequacy management is a real and challenging problem to portfolio managers in the banking industry. In this work, we investigate the problem of optimal portfolio choice of an ambiguity averse portfolio manager (AAPM) with an obligation to continuously meet her/his bank's capital adequacy requirements as specified in the BASEL III Banking Agreement. Such a problem deals with the non-linear stochastic optimal control problem whose solution is determined by means of the dynamic programming principle applied to corresponding Hamilton-Jacobi-Bellman-Isaacs (HJBI) equation. The analysis relies heavily on the stochastic modelling requiring a robust portfolio optimization approach on three categories of a bank's balance sheet items: assets, capital and liabilities. We modelled the ambiguity by means of classical dynamic programming principle and the non-linear expectation. We showed that modelling ambiguity via the classical optimal portfolio and by the Choquet expectation have completely different impacts on portfolio selection and the capital adequacy ratio by comparing the utility losses. We explored the effects of ambiguity on optimal portfolio choice and capital adequacy and demonstrated that the ambiguity aversion level decreases the optimal proportions of the risky assets while the Choquet capacity increases them. We considered the portfolio manager who wishes to maximize the expected utility of terminal wealth where only the price of the assets are available but the ambiguity parameter in the market is not observable. We used the Kalman filtering method to convert the partial information given by the observable portfolio asset and the unobservable uncertainty parameter to a full information problem. We then obtained the HJB equation of the resulting full information. We concluded that the conditional Choquet expectation gives the more realistic assumption in the derivation of the robust portfolio selection and capital adequacy.

1. Introduction

1.1 Background of the study

Optimal portfolio selection is of fundamental importance in the banking industry for investor's maximum utility from terminal wealth. The severity and the global financial crises that have been essentially remarkable in the global financial market is the catalyst of banking regulations to focus on the choice of an optimal portfolio and the capital adequacy requirements.

In the African financial markets, this problem is further compounded by low-quality data and political ambiguity. If the data's quality is low, portfolio choices are based on model parameters which are extremely difficult to estimate, creating modelling risk due to ambiguity. But as Montesano (2008) points out, ambiguity aversion seems important in financial markets, where agents are deeply concerned over the level of transparency (i.e. the reliability of the probability distribution of outcomes they refer to).

The Basel committee on Bank supervision was established to strengthen the capital regulatory framework. The Basel I contributed to the growth of securitization by assigning lower capital charges to securitized assets, thereby encouraging banks to move assets into off-balance sheet vehicles. Basel II reduced the overall demand for bank capital and consequently, its cost, leading to lower average rates for both high and low risk firms (see Repullo and Suarez (2004)). The Basel III Accord builds on the Basel I and II documents and seeks to improve the banking sector's ability to deal with financial and economic stress, improve risk management, strengthen the banks transparency (see Muller and Witbooi (2014)) and increase the amount of money banks hold as capital.

Mathematical modelling on portfolio selection dates back to the seminal work of Markowitz

(1952). In mean-variance portfolio selection model, the return on a portfolio is measured by the expected value of the portfolio return, and the associated risk is quantified by the variance of the portfolio return. This mean-variance model has had a profound impact on the economic modelling of financial markets and the pricing of assets because despite the elegance of the mean-variance model, the powerful optimization theory supporting this model, and the availability of efficient software to solve the resulting problem, the mean-variance optimization has constantly encountered doubts among investment practitioners. One of the reasons for its being doubted is that optimal portfolios are often sensitive to changes in input parameters, leading to large turnover ratios with periodic readjustment of input estimates such as the mean and covariance matrix, especially the mean (see Black and Litterman (1992)).

The first mathematical framework for this type of banking problem which is close to our interest, addressing the optimal portfolio selection and capital adequacy management problem by adopting stochastic optimization, Hamilton Jacobi Bellman (HJB) equations, and dynamic programming principles was first proposed by Mukuddem-Petersen and Petersen (2006). In a complete market setting, they minimize the capital adequacy risk by solving a nonlinear stochastic optimal control problem by means of dynamic programming principle with constant interest rate.

Chakroun and Abid (2016) used the Vacisec model as the dynamic of the interest rate by considering assets for bank account, loans, securities and liabilities as deposits, and bank capital in a complete market. An explicit risk aggregation and capital expression was provided regarding the portfolio choice and capital requirements. While control theoretic approaches can be highly useful in optimal portfolio selection and capital adequacy management, they are often constrained by the low quality of real emerging market data. Recently,

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Yi et al. (2015) published a paper on model ambiguity but in the context of insurance investments and portfolio management. In a separate effort to deal with model ambiguity under stochastic volatility, the work of Yi et al. (2015) on estimation of volatility memory parameters adapted del Moral's filters - a class of genetic algorithm to finance.

Due to the upper expectation, the probabilities are no longer additive. In this work, we address this problem by a non linear Choquet expectation described as an expected capacity in Graf (1980). The aim of this method is to derive a value function approximation (and it's iteration) close enough to the true value function while at the same time achieving computational savings.

With the fact that, only the past price movements of the portfolio assets can be the only information available in the market. This lead to the optimal portfolio selection under partial information (see Sass and Haussmann (2004)) where the expected Choquet capacity is the unobservable process. To solve the partial information-based on optimal portfolio selection problem of square integrable observable and non-observable processes we convert them to a full information problem through the utilization of the Kalman filtering technique described in Applebaum and Blackwood (2015).

To complement current phenomenological frameworks used to study optimal portfolio selection and capital adequacy management, we examine the optimal choice of an ambiguity averse portfolio manager who wants to maximize the expected utility from terminal wealth of her investors while meeting regulators' capital adequacy requirements. We use modelling, simulation, stochastic filtering and control-theoretic approaches.

1.2 Problem Statement

In this work, we consider the optimal portfolio allocation of a portfolio manager who wants to maximize the utility from terminal wealth of her investors while meeting regulators' capital adequacy requirements of her bank. In addition, she is also concerned about ambiguity in financial market models. We assume that capital allocations in the banking industry and trading in financial market securities happen in continuous time, without taxes or transaction costs.

The balance sheet items for the bank behave unpredictably from uncertain economic activities related to the evolution of treasuries, loan demand, risky and risk less investments, deposits, loans payments, borrowing and eligible regulatory capital. In most economic situations, particularly in the African financial markets, a portfolio manager cannot be confident of the probability distribution of assets in the market. This can be attributed to model ambiguity in the sense that accurate calibration of model parameters is very difficult to achieve because of poor reliability of market data. This motivated us to continue the work of Chakroun and Abid (2016) in case there is more than one equivalent martingale measure in the market.

1.3 Justification of the study

The capital management in banking industry describes why the capital requirements are the main object for the portfolio choice. It identifies what a bank should invest on and the measurements to be taken to deal with the benchmark which depends on the bank capital and the total asset portfolio.

This work is a fundamental, transformative shift in portfolio choice and capital adequacy

management, enabling accurate modelling of investor's behaviour in the context of African financial market characterized by low quality data. This work evaluates the fundamental choices and trade-offs of investments in risky assets in emerging African Financial Markets while ensuring the financial safety and soundness of the banking industry. The effects of ambiguity in regulatory policies, partial observability of key economic parameters, investor's attitude towards uncertainty and political rivalry is evaluated in details. This work covers fundamental computational issues for this and another modelling, analytical questions including the effect of ambiguity aversion on optimal portfolio choice and capital adequacy requirements, through both simulation and analytical approaches. This work presents a fundamental investment choice and decision-making framework needed at this time, as applications of computational techniques in corporate finance and capital budgeting in emerging markets surge and policy ambiguity and hidden signals from political leaders are the order of the day.

1.4   Objectives of the Study

1.4.1 General Objective. To investigate optimal portfolio strategies for investor's maximum utility from terminal wealth under strict conditions on bank capital adequacy requirement and uncertainties in emerging economic policies.

1.4.2 Specific objectives. This work addresses the following specific objectives:

1. to determine the robust portfolio that meet capital adequacy requirements using the classical approach;

2. to model the ambiguity in optimal portfolio choices by the non linear expectation method

1.5 Outline of the thesis

This work is subdivided into four chapters, chapter one introduces the study, gives the background, problem statement, objectives and the justification of the study. Chapter 2 shows what has been done in the past on Optimal Portfolio choice, chapter 3 is some of the mathematical preliminaries on Stochastic Optimal control theory and the methodology used to achieve the objectives. Chapter 4 is for results and discussions and chapter 5 is the conclusion.

2. Literature review

In this chapter, we consider the relationship between our study and previous literature. Utilizing more capital increases asset earnings and leads to higher returns on equity. In Muller and Witbooi (2014), Grant et al. modelled a Basel III compliant commercial bank that operates in a financial market consisting of a treasury security, a marketable security, and a loan regarding the stochastic interest rate in the market where the term structure of the interest rates was affine under the Vasisec and Coss-Ingersol Ross (CIR)dynamics. They came up with an investment strategy that maximizes an expected utility of the bank's asset portfolio at a future date.

In their paper, Mukuddem-Petersen and Petersen (2006) discussed optimal behaviour of a bank with respect to equity and capital adequacy risk. They have been able to find the solution of an optimal stochastic control problem that minimizes bank market and capital adequacy risks by making choices about security allocation and capital requirements, respectively. The dynamics of the lending rate was assumed to follow the geometric Brownian motion.

Lin and Li (2011) considered an optimal reinsurance-investment problem of an insurer whose surplus process follows a jump-diffusion model where the insurer transfers part of the risk due to insurance claims via a proportional reinsurance and invests the surplus in a "simplified" financial market consisting of a risk-free asset and a risky asset. They obtained explicit forms for the optimal reinsurance-investment strategy and the corresponding value function.

Witbooi et al. (2011), derived an optimal equity allocation strategy for the bank and monitored the performance of the Basel II Capital Adequacy Ratio(CAR) under the allocation strategy. In their methods, they combined the Cox-Ingersoll and Ross interest rate model and the

Cox-Huang for modelling a bank portfolio consisting of three assets: treasuries, securities and loans regarding the case of a power utility function.

An application of stochastic optimization theory to asset and capital adequacy management in banking has also been considered by Mukuddem-Petersen and Petersen (2008) who computed the dynamics of the capital adequacy ratio (CAR) in a stochastic setting by dividing regulatory bank capital (RBC) by risk weighted assets (RWAs). In their study, they demonstrated how the CAR can be optimized in terms of bank equity allocation and the rate at which additional debt and equity is raised.

Although these researches have been done, studies have been conducted in incomplete market but there have not been studies on optimal portfolio on incomplete market and capital adequacy.

Maenhout (2006) did an analysis of the optimal portfolio problem of an investor who worries about model misspecification and insists on robust decision rules when facing a mean-reverting risk premium. They presented a methodology for calculation of detection-error probabilities based on Fourier inversion of the conditional characteristic functions of the Radon–Nikodym derivatives and found that the quantitative effect of robustness is more modest than in independent and identically distribution settings.

Shen et al. (2014) provided a robust optimal hedging strategy in an incomplete market where the investor aims to minimize a function of hedging error under the worst case scenario by means of solving a min-max robust optimization problem. They applied this methodology to the asset and liability management and employed an expected shortfall hedging criterion for the value function.

In Del Vigna et al. (2011), the review of well-known simple models for portfolio selection

under ambiguity, and the computation of a number of explicit optimal portfolio rules using elementary mathematical tools has been done. In the case of a single period financial market, new results arise for an agent who is risk neutral and smoothly ambiguity averse, for a loss averse and smoothly ambiguity averse agent, for a Mean-Variance and α-Maxmin Expected Utility agent.

Yi et al. (2015) studied an ambiguity-averse insurer whose surplus process is approximated by a Brownian motion with drift, hoping to manage risk by both investing in a Black–Scholes financial market and transferring some risk to a re-insurer, but worries about uncertainty in model parameters. She chooses to find investment and reinsurance strategies that are robust with respect to this uncertainty, and to optimize her decisions in a mean-variance framework. By the stochastic dynamic programming approach, they derived expressions for a robust optimal benchmark strategy and its corresponding value function, in the sense of viscosity solutions.

Gu et al. (2017) formulated an optimal robust reinsurance investment problem. By employing the dynamic programming approach, they derived the explicit optimal robust reinsurance-investment strategy and optimal value function. By studying the portfolio allocation sensitivity to various parameters, among other things, they uncover and analyse complex behaviour resulting from asymmetry between the mean-reversion rates of the mispriced stocks. Also the analysis of various utility losses which explain the importance of ambiguity aversion, surplus-jump, mispricing, and reinsurance in their model has been defined.

The analysis of the stochastic control approach to the dynamic maximization of the robust utility of consumption and investment defined in terms of logarithmic utility and a dynamically consistent convex risk measure have been studied by Hernández-Hernández and Schied (2007). They modelled the underlying market by a diffusion process whose coefficients are

driven by an external stochastic factor process. The main results gave conditions on the minimal penalty function of the robust utility functional.

Even though, some researchers have been studying the optimal portfolio and Capital adequacy management in banking industries, none has compared the two methods, that is portfolio optimization with modelling ambiguity indirectly and portfolio optimization with modelling ambiguity directly by the Choquet capacity where the probabilities are no longer additive because of the upper expectation is the reason why we have decided to study the portfolio optimization and bank capital adequacy with the same concept as Muller and Witbooi (2014) in an incomplete market using the two methods.

The theory of capacities was started by Choquet (1954) who studied the non-additive set-functions, and tried to extract from their totality certain particularly classes, with a view to establishing for these a theory similar to the classical theory of measurability. Then Mangelsdorff and Weber (1994) tested some behavioral implications of the Choquet expected utility theory ways to assess capacities for ambiguous events.

Furthermore, Zengjing Chen (2013) with the notion of independence for random variables under upper expectations derived a law of large numbers for non-additive probabilities.

In their work, Kast et al. (2014) characterise ambiguity in the form of Choquet random walks: discrete-time binomial trees with capacities instead of exact probabilities on their branches, they described the axiomatic basis of Choquet random walks with dynamic consistency and the convergence of Choquet random walks to Choquet Brownian motion in continuous time. This leads to deformation of the standard Brownian motion where both the drift and volatility are modified.

Chen (2016) investigated three kinds of laws of large numbers for capacities with a new no-

tion of independently and identically distributed random variables for sub-linear expectations

initiated by Zengjing Chen (2013).

3. Methodology

3.1 Mathematical preliminaries on stochastic optimal contro theory

Before we start the main objectives of this work, in this chapter we highlight some of the mathematical tools which are needed. In finance, many systems are dynamic and they acquire a good control. The optimal control theory is a branch of mathematics used to control those systems in optimal way (see Gaimon (2002)).

3.1.1 Elementary concepts and definitions.

The optimal control problem considered in our work is commonly known as the maximization of the expected utility of the terminal wealth. In this section, we define the essential elements indispensable in the optimal control theory. The general references for these definitions are Pascucci and Runggaldier (2012) and Björk (2009).

In decision making, there are several situations where agents face two or more choices. The financial theory of choice uses the concept of a utility function to describe the way the decision maker (the portfolio manager) makes decisions when faced with a set of options. A utility function assigns a value to all possible choices faced by the portfolio manager. The higher the value of a particular choice, the greater the utility derived from that choice. Here, the utility function is defined as follows, and its output is a real value.

3.1.2 Definition. A utility function is a function of class C^1, $u : I \longrightarrow \mathbb{R}$ which is

  i) strictly increasing

  ii) concave

iii) in the case $a > \infty$, it holds that $\lim_{v \longrightarrow a+} u'(v) = +\infty$; in the case $a = -\infty$, u is

bounded from above.

where I denotes the real open interval $(a, +\infty)$ with a fixed constant $a \leq 0$.

Let $\mu(t, x, \nu)$ and $\sigma(t, x, \nu)$ be given functions such that

$$\mu : \mathbb{R}_+ \times \mathbb{R}^m \times \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

$$\sigma : \mathbb{R}_+ \times \mathbb{R}^m \times \mathbb{R}^n \longrightarrow \mathbb{R}^{m \times k}.$$

For a given $x_0 \in \mathbb{R}^m$, the controlled Stochastic differential equation is given as

$$dX_t = \mu(t, X_t, \nu_t)dt + \sigma(t, X_t, \nu_t)dW_t$$

$$X_0 = x_0.$$

Then, the m-dimensional X is called **state process** and it is what we try to control.

Given a subset U of $\mathbb{R}^n$, ν is the set of all progressively measurable processes

$\nu = \{v_t, t < T\}$ valued in U and the elements of ν are called **control processes**.

W is a k-dimensional Weiner process defined as follows.

3.1.3 Definition. A stochastic process W is called a Weiner process if the following conditions hold.

1. $W(0) = 0$.

2. The process W has independent increments, i.e. if $r < s < t < u$ then

   $W(u) - W(t)$ and $W(s) - W(r)$ are independent stochastic variables.

3. For $s < t$ the stochastic variable $W(t) - W(s)$ has the Gaussian distribution $N(0, t - s)$.

4. W has continuous trajectories.

In most cases, there could be constraints on the control process which requires ν to be adapted to the state process X. Moreover, at time t, the value ν_t of the control process depends on the past observed values of the state X. To obtain an adapted control process, we choose a deterministic function $g(t, x)$ such that $g : \mathbb{R}_+ \times \mathbb{R}^m \longrightarrow \mathbb{R}^n$ and $\nu_t = g(t, X_t)$. This function is called a feedback control law and $\nu(t, x)$ a control law. Then this leads us to the definition of admissible control law.

3.1.4 Definition. A control law ν is called admissible if

- $\nu(t, x) \in U$ for all $t \in \mathbb{R}_+$ and all $x \in \mathbb{R}_n$.

- For any given initial point (t, x) the Stochastic Differential Equation

$$dX_s \;\; = \;\; \mu(s, X_s, u(s, X_s))ds + \sigma(s, X_s, u(s, X_s))dW_s, \tag{3.1}$$

$$X_t \;\; = \;\; x \tag{3.2}$$

has a unique solution. The class of admissible control laws is denoted by $\mathcal{A}$.

3.1.5 Stochastic optimal control.

Maximization of expected utility

Fixing $V_0 \in \mathbb{R}$, for a predictable process π we denote by V^π the value process of a portfolio. Having defined the utility function U defined on I, our interest is the classical problem of portfolio optimization which consists of determining if it exists

$$\max_\pi \mathbb{E}[U(V^\pi)]$$

where the maximum is over the predictable process π such that

$$V^\pi \in I.$$

The control Problem has been shown to be equivalent to finding the solution to the Hamilton-Jacobi-Bellman(HJB) equation under the assumptions :

1. There exists an optimal control law $\tilde{u}$

2. The optimal value function V is regular in the sense that $V \in C^2$

Under these assumptions the following hold

1. V satisfies the HJB

$$\begin{cases} \frac{\partial V(t,x)}{\partial t} + \sup_{u \in U}\{F(t,x,u) + \mathcal{A}^u V(t,x)\} \\ \\ V(T,x) = \Phi(x), \forall x \in \mathbb{R}^n \end{cases} \tag{3.3}$$

2. For each $(t,x) \in [0,\pi] \times \mathbb{R}^n$ the supremum in the HJB is attained by $u = \tilde{u}(t,x)$.

3.1.6 Capacities and Choquet Integrals.

By Graf (1980), the capacity is defined as follows

3.1.7 Definition. Let $(X, \mathcal{F})$ be a measurable space, a map $\mathcal{C} \longrightarrow \mathbb{R}_+$ is called a capacity if the following conditions hold :

1. $\mathcal{C}(\emptyset) = 0$

2. $\forall A, B \in \mathcal{F} : \mathcal{C}(A \cup B) \leqq \mathcal{C}(A) + \mathcal{C}(B)$

3. $\forall A, B \in \mathcal{F} : A \subset B \Longrightarrow \mathcal{C}(A) \leqq \mathcal{C}(B)$

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4. For every increasing sequence $(A_n)_{n\in\mathbb{N}}$ in $\mathcal{F}$ the equality

$$\mathcal{C}\left(\cup_{n\in\mathbb{N}} A_n\right) = \lim_{n\longrightarrow\infty} \mathcal{C}(A_n) \text{ holds.}$$

5. $\forall A, B \in \mathcal{F} : \mathcal{C}(A \cap B) + \mathcal{C}(A \cup B) \leqq \mathcal{C}(A) + \mathcal{C}(B)$.

Where $\leqq$ is the canonical order relation of all the set of all capacities on $\mathcal{F}$.

Let $T \in (0, \infty)$ be a horizon time. Given a filtrated probability space $(\Omega, F, F_t, \mathbb{P})$ with $F = F_T$ and $F_t = \sigma(W_s : s \leq t)$, where $(W_t)_{0 \leq t \leq T}$ is a one-dimensional Brownian motion. Define the set

$$\mathcal{Q} = \left\{\mathbb{Q}^v : \frac{d\mathbb{Q}^v}{d\mathbb{P}} = \exp\left\{-\frac{1}{2}\int_0^T |v_s|^2 ds + \int_0^T v_s dW_S\right\}, \sup_{0\leq t\leq T} |v_s| \leq k\right\}$$

Then we have the that

$$\underline{C}(A) = \inf_{\mathbb{Q}\in\mathcal{Q}} \mathbb{Q}(A),$$

$$\overline{C}(A) = \sup_{\mathbb{Q}\in\mathcal{Q}} \mathbb{Q}(A)$$

(3.4)

$\underline{C}(A)$ and $\overline{C}(A)$ are capacities

3.2 Banking Industry model

We consider a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on a time horizon, T, with filtration,

$$\{\mathcal{F}_t\},\ t \geq 0,$$

generated by two independent standard Brownian motions $\{W_S(t), W_L(t)\},\ t \geq 0$ and $\mathbb{P}$ is a probability measure on Ω.

The bank's balance sheet assets, capital and liabilities satisfy the relation,

$$\text{total asset} = \text{total liabilities} + \text{bank capital.}$$

In particular, at any time t, the bank's stylized balance sheet is represented as follows:

$$R(t) + L(t) + S(t) \equiv B(t) + D(t) + C(t), \qquad (3.5)$$

where $R(t)$, $L(t)$, $S(t)$, $B(t)$, $D(t)$ and $C(t)$ are respectively reserves, loans, securities, deposits, borrowings and bank capital. Each of these variables is given as a function defined from $\Omega \times T \longrightarrow \mathbb{R}_+$. We further assume equal proportionality between the reserves and the borrowing plus deposit. Here, equation (3.5) becomes

$$L(t) + S(t) \equiv C(t), \qquad (3.6)$$

Following the standard practice in the banking industry (see Mukuddem-Petersen and Petersen (2006)), we put the bank securities into two categories; The first contains the riskless assets called treasuries issued by national treasuries in most countries as means of borrowing money to meet government expenditure not covered by tax revenues. The second group contains risky assets called market securities (e.g. loans, equities etc.). The dynamics of treasury is given as follows :

$$\frac{dS_0(t)}{S_0(t)} = r(t)dt. \qquad (3.7)$$

On the other hand, the dynamic of market security price is given by:

$$\frac{dS(t)}{S(t)} = (r(t) + \lambda_S)dt + \sigma_S dW_S(t) \qquad (3.8)$$

where σ_S is the security volatility and λ_S denotes the risk premium. Under the Capital Asset Pricing Model (CAPM), the risk premium can be quantified by the relation

$\lambda_S = \beta[E(R_m) - R_f]$, with $E(R_m)$ representing the market expected return, R_f the risk-free interest rate, and β the sensitivity of the expected excess asset returns to the expected excess market return. The dynamics of the loans is given as follows:

$$\frac{dL(t)}{L(t)} = (r(t) + \lambda_L)dt + \sigma_L dW_L(t), \qquad (3.9)$$

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where, $\lambda_L = \lambda_r \sigma_L + \delta$, σ_L the loan volatility, δ the default risk premium and λ_r the constant premium of interest rate risk.

Then by equations (3.7)-(3.9), the dynamics of the total asset portfolio is represented by the following equation

$$
\begin{aligned}
\frac{dX(t)}{X(t)} &= (1 - \pi_L(t) - \pi_S(t))\frac{dS_0(t)}{S_0(t)} + \pi_L(t)\frac{dL(t)}{L(t)} + \pi_S(t)\frac{dS(t)}{S(t)}, \\
&= (r(t) + \pi_L(t)\lambda_L + \pi_S(t)\lambda_S)dt + \pi_L(t)\sigma_L dW_L + \pi_S(t)\sigma_S dW_S \quad (3.10) \\
& X(0) = X_0.
\end{aligned}
$$

where $(1 - \pi_L(t) - \pi_S(t))$, $\pi_L(t)$ and $\pi_S(t)$ are the proportions, invested in treasuries, loans and market securities respectively.

In the Basel III accord, the regulatory capital can be divided into Tier 1 and Tier 2 capital (see Eubanks (2010)). Then the bank capital at time $t \geq 0$ is given by;

$$
C(t) = C_{T_1}(t) + C_{T_2}(t).
$$

The Tier 1 capital which describes the capital adequacy of a bank includes equity capital $E(t)$ and retained earnings. Tiers 2 capital includes subordinated debt $S_D(t)$, limited life preferred stocks, loans losses reserves and good will. Since the nature of retained earnings, life preferred stocks and loan-loss reserves are not dynamic, we do not consider these as active constituents of bank capital. Thus, the total bank capital is given as:

$$
C(t) = E(t) + S_D(t).
$$

Assuming that the market value of the sum of subordinate debt is given by

$$
S_D = S_D(0)e^{\int_0^t r(s)ds}. \quad (3.11)
$$

In what follows, the bank is assumed to hold capital in $n + 1$ categories, one of which is risk free and corresponds to subordinated debt, and n categories for bank equity when each one

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of them is modelled as follows:

$$\frac{dE(t)}{E(t)} = (r(t) + \lambda_E(t))dt + \sigma_E dW_E(t) \tag{3.12}$$

where λ_E represents the market price of risk and σ_E the equity volatility. Hence, the total bank-capital dynamics becomes

$$
\begin{aligned}
\frac{dC(t)}{C(t)} &= \sum_{i=1}^{n} \pi_i(t)\frac{dE(t)}{E(t)} + \Big(1 - \sum_{i=1}^{n} \pi_i(t)\Big)\frac{dS_D(t)}{S_D(t)} - \rho X(t)dt \\
&= (r(t) + \pi^{tp}(t)\lambda_E)dt + \pi^{tp}(t)\sigma_E dW_E(t) - \rho X(t)dt
\end{aligned}
$$

where $\pi^{tp}(t)$ is the transpose vector of the optimal proportions invested in loans and securities. At time t, the bank capital is converted into loans and securities at the rate of $\rho X(t) = \rho X(t)dt$ for a constant ρ.

Let $X_r(t)$ denotes the total risk-weighted assets defined by placing each of the on-and-off balance sheet items into a risk category such that the Capital Adequacy Ratio (CAR) is defined as,

$$\text{CAR} = \frac{C(t)}{X_r(t)}. \tag{3.13}$$

So, one of the portfolio manager's problems is to consider the capital constraints such that CAR $\geq \rho$.

In most economic situations, particularly in the African financial markets, a bank manager cannot be confident of the probability distribution of assets in the market. This can be attributed to model ambiguity in the sense that accurate calibration of model parameters is very difficult to achieve because of poor reliability of market data. This means that there is more than one equivalent martingale measure in the market. Let $\mathbb{P}$ be a reference probability measure and $\mathcal{Q}$ denote the set of probability measures $\mathbb{Q}$ such that $\mathbb{Q} \sim \mathbb{P}$. Then

the robust strategy has to deal with this by following the approach of an equivalent martingale measure $\mathbb{Q}$ to $\mathbb{P}$ (see Hernández-Hernández and Schied (2007))to penalize each such model with a penalty $\varphi(\mathbb{Q})$. Now, for a bank with a strictly initial asset value $X(0)$, and shareholders enjoying a power utility function U, the bank manager faces an additional problem of maximizing share-holder's utility from terminal wealth with model ambiguity. This leads to a search for robust optimal strategies of the following form;

$$\sup_{\pi(t)\in\mathcal{A}} \inf_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_t^{\mathbb{Q}(\pi)}\Big[U(X_T) + \varphi(\mathbb{Q}(\pi))\Big] \tag{3.14}$$

where, $\mathcal{A}$ is the set of control processes for an ambiguity-neutral regulator in a given market and $E_t^{\mathbb{Q}}[\cdot] = E^{\mathbb{Q}}[\cdot|F_t]$ represents the conditional expectation under the probability measure $\mathbb{Q}$.

Approach and Strategies: In order to tackle the ambiguity, the regulators has to consider some alternatives to $\mathbb{P}$. Every alternative model is characterized by a stochastic process θ and the associated probability measure $\mathbb{Q}$, which is equivalent to the reference measure $\mathbb{P}$ (see Gu et al. (2017)). Let us denote this class of probability measures by $\mathcal{Q} = \{\mathbb{Q} : \mathbb{Q} \sim \mathbb{P}\}$. The property of the Radon-Nikodym Theorem (see Chorro (2009)) is used to find the set of equivalent martingale measures to the reference probability measure. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space satisfying the usual conditions and $\mathbb{Q}$ be another probability measure on $(\Omega, \mathcal{F}, \mathbb{Q})$ under the assumption that $\mathbb{Q} \ll \mathbb{P}$, then there exist a non negative random variable $\frac{d\mathbb{Q}}{d\mathbb{P}} = Z$ and we call Z the Radon-Nikodym derivative of $\mathbb{Q}$ with respect to $\mathbb{P}$. Then, by the Cameron-Martin-Girsanov theorem (see Björk (2009)), we consider, an adapted process $\theta(t) = (\theta_L(t), \theta_S(t))$ such that for every $\mathbb{Q} \in \mathcal{Q}$, $\frac{d\mathbb{Q}}{d\mathbb{P}} = Z(t)$, where

$$Z(t) = e^{\int_0^t \theta_L(u)dW_L(u) + \int_0^t \theta_S(u)dW_S(u) - \frac{1}{2}\int_0^t (\theta_L^2(u) + \theta_S^2(u))du}$$

and $Z(t)$ is a positive (W_L, W_S) martingale under $\mathbb{P}$ for $0 \leq u < t < T$.

To ensure the martingale property, we assume that $\theta(t)$ satisfies the bounded condition, that is there exists a constant $c > 0, \; \forall t \in [0, T]$ such that

$$\| \theta(t) \|^2 < c \text{ a.s.} \tag{3.15}$$

Furthermore, by the Girsanov's theorem, to achieve the model uncertainty we allow the drift parameters in the incomplete market to be undetermined. This means that we add the drift term $\theta(t)$ to the independent standard Brownian motions (dW_L, dW_S) under $\mathbb{P}$, and we have

$$dW_L^Q = dW_L + \theta_L(t)dt \tag{3.16}$$

$$dW_S^Q = dW_S + \theta_S(t)dt \tag{3.17}$$

two standard independent Brownian motion under $\mathbb{Q} \in \mathcal{Q}$.

3.3   Classical Optimal portfolio choice problem

A myriad of attributes has been linked to investors in African financial market. Some are ambiguity averse due to complex information structures while others can be regarded as either ambiguity neutral or seeking Epstein and Miao (2003). We want to determine a unique strategy π^* from the set of admissible strategies $\mathcal{A}$ and a unique measure $\mathbb{Q} \in \mathcal{Q}$ such that the portfolio manager can maximize the utility from terminal wealth of her ambiguity aversed investors. From equations (3.16) and (3.17) into (3.10), we can write the dynamics of the total asset portfolio under the alternative model $\mathbb{Q}$:

$$\frac{dX(t)}{X(t)} = [r(t) + \pi_L(t)(\lambda_L - \sigma_L(t)\theta_L) + \pi_S(t)(\lambda_S - \sigma_S\theta_S(t)]dt + \pi_L(t)\sigma_L dW_L^{\mathbb{Q}} + \pi_S(t)\sigma_S dW_S^{\mathbb{Q}}$$

(3.18)

A strategy, π, from the set;

$$\mathcal{A} = \left\{ \pi(\cdot) = \pi(t)_{t \in [0,T]}, \mathcal{F} - \text{adapted.} \right\}$$

is admissible if

- $X_t \geq K$ for a constant $K > 0, 0 \leq t \leq T$;

- $\mathbb{E}^{Q^*}[\int_0^t (\pi_S \sigma_s)^2 + (\pi_L \sigma_L)^2 dt] < \infty$;

- $\forall (t,x) \in [0,T] \times \mathbb{R}$, equation (3.10) has a unique solution $\{X(t)\}_{t \in [0,T]}$ with $\mathbb{E}^{Q^*}[U(X(t)] < \infty$ where Q^* is the model worst-case-scenario probability measure.

In our work, we propose that the worst case scenario measure $Q^*(\pi)$ is the required unique measure $Q^*(\pi) \in \mathcal{Q}$ that satisfies the above conditions. To determine this measure explicitly and the corresponding value function $V(t,x)$, we set up and solve a stochastic optimal control problem through the dynamic programming approach in a Brownian motion setting. In particular, the following expression holds

$$V(t,x) = \mathbb{E}_t^{Q^*(\pi^*)}[U(X_T)] = \sup_{\pi \in \mathcal{A}} \inf_{\mathbb{Q} \in \mathcal{Q}} \left\{ \mathbb{E}_t^{\mathbb{Q}(\pi)} \left[U(X_T) + \varphi(\mathbb{Q}(\pi)) \right] \right\}$$

(3.19)

We now solve the corresponding Hamilton-Jacobi-Bellman-Isaacs (HJBI) Forsyth and Labahn (2007) equation for a power utility function over the time horizon T. That is:

$$U(X) = \frac{X^{1-\gamma}}{1-\gamma}, \ 0 < \gamma < 1.$$

(3.20)

and the boundary condition given by $V(T, x) = U(x)$. This implies that

$$\sup_{\pi \in \mathcal{A}} \inf_{(\theta_L, \theta_S) \in \theta(t)} \{\triangle^\pi V + \varphi(\theta(t))\} = 0 \tag{3.21}$$

where the differential operator $\triangle^\pi$ is computed as follows;

$$\triangle^\pi V = V_t + V_x x \left[r(t) + \pi_L(t)(\lambda_L - \sigma_L(t)\theta_L(t)) + \pi_S(t)(\lambda_S - \sigma_S\theta_S(t)) \right] + \frac{1}{2} V_{xx} x^2 \left[(\pi_L(t)\sigma_L)^2 + (\pi_S(t)\sigma_S)^2 \right] \tag{3.22}$$

Specifically, since any $\mathbb{Q} \in \mathcal{Q}$ is defined with respect to the pair $(\theta_L(t), \theta_S(t))$ we define a map Q^* taking any strategy π to $(\theta_L(t), \theta_S(t))$. Then the penalty function will be determined by a map taking any of the strategy satisfying (3.15). That is

$$\varphi(\theta(t)) = \frac{1}{2\phi} \left(\theta_L^2(t) + \theta_S^2(t) \right) \tag{3.23}$$

Where ϕ is the preference parameter which governs the ambiguity aversion.

By Maenhout (2004), we impose that the preference parameter $\phi > 0$ depends on the state variable (the current wealth process $V(t, x)$) in order to ensure that the penalty function is reasonable. We then choose the preference parameter ϕ to be given by $\phi = \frac{\alpha}{(1-\gamma)V(t,x)} > 0$. Where $\alpha > 0$ indicates the portfolio manager's ambiguity aversion level.

3.3.1 Proposition. The solution of (3.21) is given by $V(t, x) = f(t)\frac{x^{1-\gamma}}{1-\gamma}$. Where the boundary conditions $f(T) = 1$, $f(t) = e^{\beta(T-t)}$ with $\beta = (1 - \gamma)\left[r + \frac{\left(\frac{\lambda_L^2}{\sigma_L^2} + \frac{\lambda_S^2}{\sigma_S^2} \right)}{2\left(\alpha + \gamma\right)} \right]$ and the optimal investment strategy given by

$$\pi_L^*(t) = \frac{\lambda_L}{\sigma_L^2 \left(\alpha + \gamma \right)} \tag{3.24}$$

$$\pi_S^*(t) = \frac{\lambda_S}{\sigma_S^2 \left(\alpha + \gamma \right)} \tag{3.25}$$

Proof. See Apendix A □

3.4 Optimal portfolio choice with Choquet expectation

3.4.1 Model ambiguity by Choquet Brownian motion.

In this case, instead of modelling ambiguity indirectly by the penalty function in (3.14), we have the robust optimal strategy in the form

$$\sup_{\pi(t)\in\mathcal{A}} \inf_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_t^{\mathbb{Q}(\pi)}\Big[U(X_T)\Big] \tag{3.26}$$

from this, we can see that

$$
\begin{aligned}
\inf_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_t^{\mathbb{Q}(\pi)}\Big[U(X_T)\Big] &= \inf_{\mathbb{Q}\in\mathcal{Q}} \int_\Omega U(X_T)d\mathbb{Q}\\
&= \int_\Omega U(X_T)\inf_{\mathbb{Q}\in\mathcal{Q}} d\mathbb{Q}\\
&= \int_\Omega U(X_T)d\{\inf_{\mathbb{Q}\in\mathcal{Q}}\mathbb{Q}\}
\end{aligned}
$$

If the portfolio manager assumes that there exist a capacity c such that $c(\cdot) = \inf_{\mathbb{Q}\in\mathcal{Q}}\mathbb{Q}(\cdot)$, then we have

$$
\begin{aligned}
\inf_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_t^{\mathbb{Q}(\pi)}\Big[U(X_T)\Big] &= \int_\Omega U(X_T)dc\\
&= \mathbb{E}_t^c\Big[U(X_T)\Big]
\end{aligned}
$$

which implies that the value function is now given by

$$V(t,x) = \sup_{\pi(t)\in\mathcal{A}} \mathbb{E}_t^c\Big[U(X_T)\Big] \tag{3.27}$$

where $\mathbb{E}_t^c$ is the Choquet expectation.

In order to find the Choquet expectation, we describe the ambiguity aversion by a binomial tree as in figure 3.1 where we have the joint capacity on each asset.
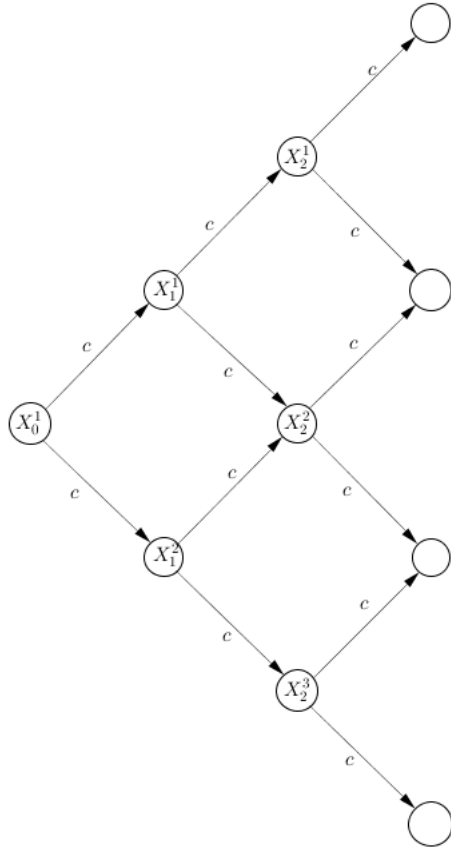
Figure 3.1: Binomial tree for a Choquet random walk.

As proven in Kast et al. (2014), with the choquet random walk in continuous time $t \in [0, T]$,

$$dW_L(t) = \mu dt + sdB_L(t)$$

$$dW_S(t) = \mu dt + sdB_S(t)$$

where $\mu = 2c - 1$, $s^2 = 4c(1 - c)$. The attitude towards ambiguity with the Choquet Brownian

motion modifies the drift and the volatility terms of the total asset as follows;

$$
\begin{aligned}
\frac{dX(t)}{X(t)} &= (r(t) + \pi_L \lambda_L + \pi_S \lambda_S)dt + \pi_L \sigma_L(\mu dt + sdB_L(t)) + \pi_S \sigma_S\Big(\mu dt + sdB_S(t)\Big) \\
&= \Big(r(t) + \pi_L \lambda_L + \pi_S \lambda_S + \pi_L \sigma_L \mu + \pi_S \sigma_S \mu\Big)dt + \pi_L \sigma_L sdB_L(t) + \pi_S \sigma_S sdB_S(t)
\end{aligned}
$$

In order to solve (3.27), we solve the corresponding Jacobi-Bellman equation

$$\sup_{\pi \in \mathcal{A}} \{\triangle^{\pi} V\} = 0 \qquad (3.28)$$

where the differential operator $\triangle^{\pi} V$ is computed as;

$$\triangle^{\pi} V = V_t + V_x x \Big[r(t) + \pi_L \lambda_L + \pi_S \lambda_S + \pi_L \sigma_L \mu + \pi_S \sigma_S \mu \Big] + \frac{1}{2} V_{xx} x^2 \Big[(\pi_L(t) \sigma_L s)^2 + (\pi_S(t) \sigma_S s)^2 \Big]$$

We therefore have the following proposition as the solution

3.4.2 Proposition. The solution of (3.28) is given by $V(t,x) = \varphi(t) \frac{x^{1-\gamma}}{1-\gamma}$. Where the boundary conditions $\varphi(T) = 1$, $\varphi(t) = e^{\eta(T-t)}$ with

$$\eta = (1-\gamma) \left[r(t) - \left(1 + \frac{\gamma}{2}\right) \left(\frac{(\lambda_L + \sigma_L \mu)^2}{\sigma_L^2 s^2} + \frac{(\lambda_S + \sigma_S \mu)^2}{\sigma_S^2 s^2} \right) \right]$$

and the optimal investment strategy given by

$$\pi_L^*(t) = \frac{\lambda_L + \sigma_L \mu}{\gamma \sigma_L^2 s^2} \qquad (3.29)$$

$$\pi_S^*(t) = \frac{\lambda_S + \sigma_S \mu}{\gamma \sigma_S^2 s^2} \qquad (3.30)$$

Proof. Assuming that $V(t,x) = \varphi(t) \frac{x^{1-\gamma}}{1-\gamma}$, then we have

$$V_t = \varphi'(t) \frac{x^{1-\gamma}}{1-\gamma} \qquad (3.31)$$

$$V_x = \varphi(t) x^{-\gamma} \qquad (3.32)$$

$$V_{xx} = -\gamma \varphi(t) x^{-1-\gamma} \qquad (3.33)$$

Differentiating (3.28) with respect to π_L and π_S gives

$$xV_x(\lambda_L + \sigma_L \mu) + V_{xx} x^2 \pi_L \sigma_L^2 s^2 = 0$$

$$xV_x(\lambda_S + \sigma_S \mu) + V_{xx} x^2 \pi_S \sigma_S^2 s^2 = 0$$

Which implies that

$$
\begin{aligned}
\pi_L &= -\frac{xV_x(\lambda_L + \sigma_L\mu)}{V_{xx}x^2\sigma_L^2 s^2} \\
&= \frac{\lambda_L + \sigma_L\mu}{\gamma\sigma_L^2 s^2} \\
\pi_S &= -\frac{xV_x(\lambda_S + \sigma_S\mu)}{V_{xx}x^2\sigma_S^2 s^2} \\
&= \frac{\lambda_S + \sigma_S\mu}{\gamma\sigma_S^2 s^2}
\end{aligned}
$$

Replacing π_L and π_S in (3.28) yields

$$
\varphi'(t)\frac{x^{1-\gamma}}{1-\gamma} + x\varphi(t)x^{-\gamma}\left[r(t) - \lambda_L\frac{\lambda_L + \sigma_L\mu}{\sigma_L^2 s^2} - \lambda_S\frac{\lambda_S + \sigma_S\mu}{\sigma_S^2 s^2} - \sigma_L\mu\frac{\lambda_L + \sigma_L\mu}{\sigma_L^2 s^2} - \sigma_S\mu\frac{\lambda_S + \sigma_S\mu}{\sigma_S^2 s^2}\right]
$$

$$
-\frac{1}{2}\gamma\varphi(t)x^{1-\gamma}\left[\sigma_L^2 s^2\left(\frac{\lambda_L + \sigma_L\mu}{\sigma_L^2 s^2}\right)^2 + \sigma_S^2 s^2\left(\frac{\lambda_S + \sigma_S\mu}{\sigma_S^2 s^2}\right)^2\right] = 0
$$

$$
\implies \varphi'(t)\frac{1}{1-\gamma} + \varphi(t)\left[r(t) - \lambda_L\frac{\lambda_L + \sigma_L\mu}{\sigma_L^2 s^2} - \lambda_S\frac{\lambda_S + \sigma_S\mu}{\sigma_S^2 s^2} - \sigma_L\mu\frac{\lambda_L + \sigma_L\mu}{\sigma_L^2 s^2} - \sigma_S\mu\frac{\lambda_S + \sigma_S\mu}{\sigma_S^2 s^2}\right]
$$

$$
-\frac{1}{2}\gamma\varphi(t)\left[\sigma_L^2 s^2\left(\frac{\lambda_L + \sigma_L\mu}{\sigma_L^2 s^2}\right)^2 + \sigma_S^2 s^2\left(\frac{\lambda_S + \sigma_S\mu}{\sigma_S^2 s^2}\right)^2\right] = 0
$$

$$
\implies \varphi'(t)\frac{1}{1-\gamma} + \varphi(t)\left[r(t) - \frac{(\lambda_L + \sigma_L\mu)^2}{\sigma_L^2 s^2} - \frac{(\lambda_S + \sigma_S\mu)^2}{\sigma_S^2 s^2}\right]
$$

$$
-\frac{1}{2}\gamma\varphi(t)\left[\frac{(\lambda_L + \sigma_L\mu)^2}{\sigma_L^2 s^2} + \frac{(\lambda_S + \sigma_S\mu)^2}{\sigma_S^2 s^2}\right] = 0
$$

$$
\implies \varphi'(t) + \varphi(t)(1-\gamma)\left[r(t) - \left(1 + \frac{\gamma}{2}\right)\left(\frac{(\lambda_L + \sigma_L\mu)^2}{\sigma_L^2 s^2} + \frac{(\lambda_S + \sigma_S\mu)^2}{\sigma_S^2 s^2}\right)\right] = 0
$$

$$
\implies \varphi'(t) + \eta\varphi(t) = 0
$$

With

$$
\eta = (1-\gamma)\left[r(t) - \left(1 + \frac{\gamma}{2}\right)\left(\frac{(\lambda_L + \sigma_L\mu)^2}{\sigma_L^2 s^2} + \frac{(\lambda_S + \sigma_S\mu)^2}{\sigma_S^2 s^2}\right)\right]
$$

This complete the proof. $\qquad\square$

3.4.3 Kalman Filter of conditional Choquet expectation.

Let F_t denotes the filtration generated by observing X defined by the $\sigma-\text{field}$

$$F_t = \sigma\{X_s : s \leq t\}$$

which is all the information available to the observer at time $t \leq T$. For any square integrable function $g : \mathbb{R} \longrightarrow \mathbb{R}$, the posterior moment is $\hat{c}_t = \mathbb{E}\{g(c_t)|F_t\}$, we want to know under which measure given by new parameter c, our observation would have been the most probable.

We assume that the asset capacity is normally distributed and its initial distribution has mean c_0 and variance v_0. During each time interval, the portfolio manager observes the loans and securities which are correlated with the capacity c_t. For the filtering problem in the context of square integrable processes, we allow the unobservable process to be given by the dynamic

$$dc_t = \theta(\mu - c_t)dt + \sigma dW_t \tag{3.34}$$

The positive parameters σ, θ and the long run mean drift level μ are considered to be known. We denote by $\hat{c}(t) = \mathbb{E}(c(t)|F_t)$ and $v(t) = \mathbb{E}[c(t) - \hat{c}(t))^2|F_t]$ respectively the expectation and the variance of the capacity at time t conditional on the available information up to time t. By observing the asset prices, the portfolio manager can collect its beliefs on the value of the capacity.

With normality assumptions, from theorem 12.1 of Liptser and Shiryaev (2010) it follows that the instantaneous changes in the expected estimated capacity $\hat{c}(t)$ and the instantaneous changes in the variance of estimated capacity $dv(t)$ are given by the following equations

$$d\hat{c}_t = \theta(\mu - \hat{c}_t)dt + \frac{2v(t)}{s}[dX(t) - (r + \pi_L(\lambda_L - \sigma_L) + \pi_S(\lambda_S - \sigma_S) + 2(\pi_L\sigma_L + \pi_S\sigma_S)\hat{c}(t)dt)] \tag{3.35}$$

$$\dot{v}(t) = -2\theta v(t) + \sigma^2 - \frac{4v(t)^2}{s^2} \qquad (3.36)$$

In (3.35), the change in capacity, $d\hat{c}(t)$ is equal to the estimated expected capacity at time t plus the correction term. This term brings the weighting value and instantaneous change due to the observation of X over the period $[t, t+dt]$. This implies that the portfolio manager updates $c(t)$ by the component $d\nu_t = dX(t) - (r + \pi_L(\lambda_L - \sigma_L) + \pi_S(\lambda_S - \sigma_S) + 2(\pi_L \sigma_L + \pi_S \sigma_S)\hat{c}(t)dt)$ weighted by its corresponding uncertainty

$$w(t) = \frac{2v(t)}{s}$$

The weight $w(t)$ in (3.35), determines how much of the new information contained in the updating of $c(t)$.

When there is low quality data (high value of $w(t)$), little information can be known and then there is not much change in $c(t)$. If the portfolio manager is less confident of the current estimate of the higher value of the variance, more information can be obtained and in this case, the portfolio manager add more weights on the obtained information to conclude its beliefs.

The first two terms of (3.36) imply that the unobservable variation of $c(t)$ over the period $[t, t+dt]$ and the last term shows the reduction in variance when more information rises up. This means that, the higher the quality of data, the more the portfolio manager learns about the current value of the capacity. Given a power utility function, we define the value function of the portfolio problem with partial information as

$$V(t,x) = \sup_{\pi(t) \in \mathcal{A}} \mathbb{E}\{U(X_T)|X_t = x\} \qquad (3.37)$$

29

(3.37) is a non-Markovian formulation of the problem. However, the Kalman-filter fully parameterizes the conditional distribution $c_t|F_t$, and (3.37) turns out to be a Markov control problem, meaning the value function is a deterministic function of $(X_t, \hat{c}_t)$.

For the partial information case, the control problem can be written in its Markovian formulation as follows:

$$V(t,x) = V(t,x,c) = \sup_{\pi(t) \in \mathcal{A}} \mathbb{E}\left\{U(X_T)|X_0 = x, \hat{c}_0 = c_0, v_0 = e\right\}. \tag{3.38}$$

3.4.4 Proposition. Suppose the filtering distribution Π_t belongs to a Hilbert space $\mathcal{H}$, then, the value function, $V(t, \Pi, x)$ is the unique viscosity solution of

$$V_t + \mathcal{L}V = 0 \tag{3.39}$$

and is bounded and locally Lipschitz with respect to the Hilbert norm. Also, the test function $V(t, \Pi_t, x)$ generally satisfies;

$$\mathcal{L}V = \frac{1}{2}\mathrm{tr}\left(\begin{bmatrix} \sigma_L\pi_L sx & \sigma_S\pi_S sx \\ \frac{2v(t)}{s} & 0 \end{bmatrix}\begin{bmatrix} \sigma_L\pi_L sx & \sigma_S\pi_S sx \\ \frac{2v(t)}{s} & 0 \end{bmatrix}^T D^2V\right) + \langle\theta(\mu - \hat{c}), DV\rangle$$
$$+ \left[r + \pi_L(\lambda_L - \sigma_L) + \pi_S(\lambda_S - \sigma_S) + 2(\pi_L\sigma_L + \pi_S\sigma_S)\hat{c}(t)\right]x\partial xV \tag{3.40}$$

where D is the Fréchet derivative and $\langle., .\rangle$ represents the inner product in the Hilbert space $\mathcal{H}$.

Then we have

$$
\begin{bmatrix} \sigma_L \pi_L sx & \sigma_S \pi_S sx \\ \frac{2v(t)}{s} & 0 \end{bmatrix} \begin{bmatrix} \sigma_L \pi_L sx & \sigma_S \pi_S sx \\ \frac{2v(t)}{s} & 0 \end{bmatrix}^T D^2 V = \begin{bmatrix} \left[(\sigma_L \pi_L)^2 + (\sigma_S \pi_S)^2 \right] s^2 x^2 & \frac{2\sigma_L \pi_L sv(t)}{s} x \\ \frac{2\sigma_L \pi_L sv(t)}{s} x & \frac{4v(t)^2}{s^2} \end{bmatrix} \begin{bmatrix} V_{xx} & V_{x\hat{c}} \\ V_{x\hat{c}} & V_{\hat{c}\hat{c}} \end{bmatrix}
$$

$$
= \begin{bmatrix} \left[(\sigma_L \pi_L)^2 + (\sigma_S \pi_S)^2 \right] s^2 x^2 V_{xx} + 2\sigma_L \pi_L v(t) x V_{x\hat{c}} & [(\sigma_L \pi_L)^2 + (\sigma_S \pi_S)^2] s^2 x^2 V_{x\hat{c}} + 2\sigma_L \pi_L v(t) x V_{\hat{c}\hat{c}} \\ 2\sigma_L \pi_L v(t) x V_{xx} + \frac{4v(t)^2}{s^2} V_{x\hat{c}} & 2\sigma_L \pi_L v(t) x V_{x\hat{c}} + \frac{4v(t)^2}{s^2} V_{\hat{c}\hat{c}} \end{bmatrix}
$$

$$\tag{3.41}$$

$$
\begin{aligned}
\langle \theta(\mu - \hat{c}), DV \rangle &= \begin{bmatrix} \theta(\mu - \hat{c}) & 0 \end{bmatrix} \begin{bmatrix} V_{\hat{c}} \\ V_x \end{bmatrix} \\
&= \theta(\mu - \hat{c}) V_{\hat{c}} \left[r + \pi_L(\lambda_L - \sigma_L) + \pi_S(\lambda_S - \sigma_S) + 2(\pi_L \sigma_L + \pi_S \sigma_S)\hat{c}(t) \right] x \partial x V \\
&= \left[r + \pi_L(\lambda_L - \sigma_L) + \pi_S(\lambda_S - \sigma_S) + 2(\pi_L \sigma_L + \pi_S \sigma_S)\hat{c}(t) \right] x V_x
\end{aligned}
$$

Therefore the corresponding HJB is

$$
\sup_{\pi \mathcal{A}} \left\{ V_t + \theta(\mu - \hat{c})V_c + \frac{1}{2}(\pi_L^2 \sigma_L^2 + \pi_S^2 \sigma_S^2)s^2 x^2 V_{xx} + \left[r + \pi_L(\lambda_L - \sigma_L) + \pi_S(\lambda_S - \sigma_S) \right. \right.
$$
$$
\left. \left. + 2(\pi_L \sigma_L + \pi_S \sigma_S)\hat{c}(t) \right] x V_x + 2\sigma_L \pi_L v(t) x V_{x\hat{c}} + \frac{2v(t)^2}{s^2} V_{\hat{c}\hat{c}} \right\} = 0 \tag{3.42}
$$

4. Results and discussion

4.1 Classical optimal portfolio choice problem

From proposition 3.3.1, the optimal investment strategy is the classical optimal solution for Merton (1971) with the risk-aversion adjustment replaced by $\alpha + \gamma$. This shows that the optimal investment strategy depends on risk aversion measure γ and the ambiguity aversion level α. If the portfolio manager does not consider ambiguity (that is $\alpha = 0$), the optimization problem (3.19) becomes an optimization problem for ambiguity neutral.

4.1.1 Calibration and interpretation of the ambiguity averse level α.

In order to predict the quantitative effect of robustness on the portfolio choice given by (3.24) and (3.25), we suggest the calibration of the parameter α. From (3.24) and (3.25) we have

$$
\begin{aligned}
\theta_L^*(t) &= \sigma_L \phi x V_x \pi_L(t) \\
&= \frac{\alpha \lambda_L}{\sigma_L(\alpha + \gamma)}
\end{aligned}
\tag{4.1}
$$

$$
\begin{aligned}
\theta_S^*(t) &= \sigma_S \phi x V_x \pi_L(t) \\
&= \frac{\alpha \lambda_S}{\sigma_S(\alpha + \gamma)}
\end{aligned}
\tag{4.2}
$$

Table 4.1 reports the optimal investments strategies allocated to the total assets and the worst-case drifts from various values of the ambiguity level α.

Table 4.1: Optimal portfolio shares and worst-case drifts

| | $\gamma = 0.1$ | | | | | $\gamma = 0.5$ | | | | |
α	π_L	π_S	$1 - \pi_L - \pi_S$	θ_L	θ_S	π_L	π_S	$1 - \pi_L - \pi_S$	θ_L	θ_S
0	23.4375	15.4321	-37.8696	0	0	4.6875	3.0864	-6.7739	0	0
0.1	11.7188	7.7160	-18.4348	0.1875	0.1389	3.9063	2.5720	-5.4783	0.0625	0.0463
0.4	4.6875	3.0864	-6.7739	0.3000	0.2222	2.6042	1.7147	-3.3188	0.1667	0.1235
0.5	3.9063	2.5720	-5.4783	0.3125	0.2315	2.3438	1.5432	-2.887	0.1875	0.1389
0.7	2.9297	1.9290	-3.8587	0.3281	0.2430	1.9531	1.2860	-2.2391	0.2187	0.1620
0.8	2.6042	1.7147	-3.3188	0.3333	0.2469	1.8029	1.1871	-1.9901	0.2308	0.1709
1	2.1307	1.4029	-2.5336	0.3409	0.2525	1.5625	1.0289	-1.5914	0.2500	0.1852
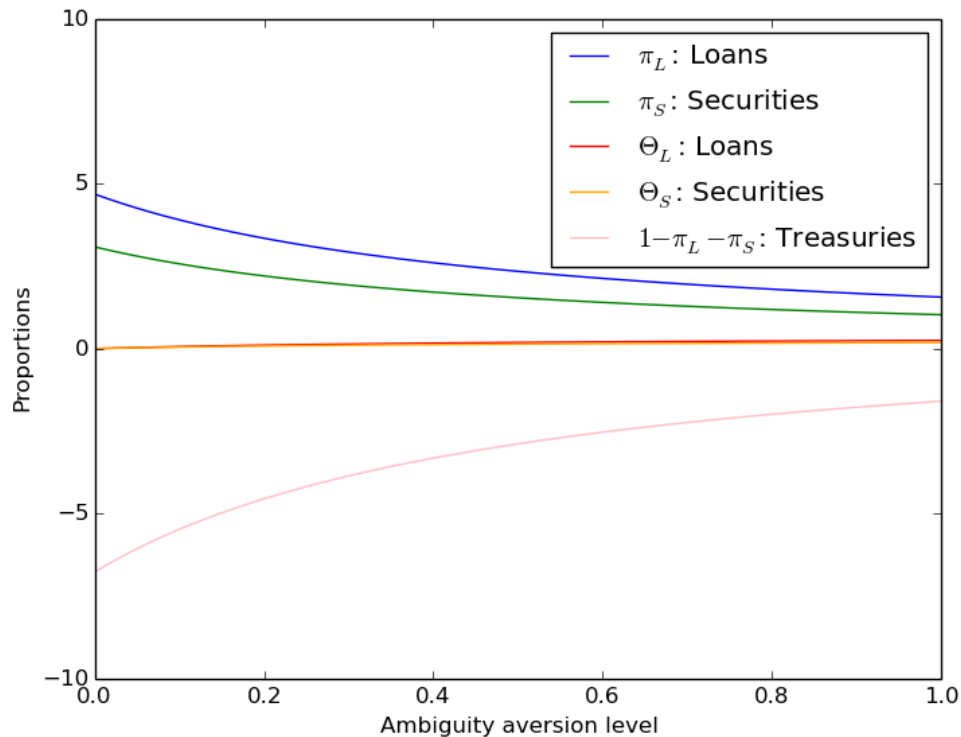
Figure 4.1: Optimal Investment Portfolio Weights and worst-case drifts

Setting the parameters estimated in Chakroun and Abid (2016), the observation of table

4.1 and figure 4.1 shows that the preference for robustness decreases the optimal portfolio weight because by looking at the values of π_L and π_S we see that the more the ambiguity aversion level α increases, the more the the optimal portfolio weight decreases. Unlikely for the worst-case drifts we see that the parameters θ_L and θ_S increases with the ambiguity aversion level. If the parameter $\alpha = 0$, then the worst-case drifts vanishes and the Ambiguity Averse Portfolio reduces to the Ambiguity Neutral.

By figure 4.1 for the risk-aversion $\gamma = 0.5$ and we are able to see the behaviour of the optimal investment weights and the worst case drifts in response to the different values of the ambiguity averse level α and we see that the curves of the drifts remain constants as α tends to infinity, but since the penalty term will vanish at that time, the Ambiguity Averse Portfolio manager has to look for another alternative model.

Now that we have been able to find out the optimal values of the parameters for the wealth process, in the following section we use them to find the capital adequacy.

4.1.2 Bank Capital Adequacy ratio.

By the Basel III accord, the risk weighted asset is given as

$$
\frac{dX_r(t)}{X_r(t)} = \left[0.5\pi_L(t)\Big(r(t) + \lambda_L - \sigma_L\theta_L(t)\Big) + 0.2\pi_S(t)\Big(r(t) + \lambda_S - \sigma_S\theta_S(t)\Big)\right]dt
$$
$$
+ 0.5\sigma_L\pi_L(t)dW_L^Q + 0.2\sigma_S\pi_S(t)dW_S^Q
$$

By (3.13) and Itô formula, the derivation of the capital adequacy ratio will be given as

$$
\begin{aligned}
d\frac{C(t)}{X_r(t)} &= \frac{dC(t)}{X_r(t)} - \frac{C(t)dX_r(t)}{X_r^2(t)} + \frac{C(t)(dX_r(t))^2}{X_r^3(t)} - \frac{dX_r(t)dC(t)}{X_r^2(t)} \\
&= \frac{C(t)}{X_r(t)}\Big[(r(t) + \pi^{tp}(t)\lambda_E)dt + \pi^{tp}(t)\sigma_E dW_E(t)\Big] - \rho\frac{X(t)}{X_r(t)}dt \\
&\quad - \frac{C(t)}{X_r(t)}\Big[0.5\pi_L(t)(r(t) + \lambda_L - \sigma_L\theta_L(t)) + 0.2\pi_S(t)(r(t) + \lambda_S - \sigma_S\theta_S(t))\Big]dt \\
&\quad + \frac{C(t)}{X_r(t)}\Big[(0.5\pi_L\sigma_L)^2 + (0.2\pi_L\sigma_L)^2\Big]dt - \frac{C(t)}{X_r(t)}\Big[0.5\pi_L(t)\sigma_L dW_L^Q + 0.2\pi_S(t)\sigma_S dW_S^Q\Big]
\end{aligned}
$$

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$$= \left[C_r(t)(a_1 - b_1) - c_1 \right] dt - C_r(t) \left[a_2 dW_L^Q + b_2 dW_S(t) - c_2 dW_E(t) \right]$$

Where

$$a_1 = r(t) + \pi^{tp}(t)\lambda_E$$

$$b_1 = 0.5\pi_L(t)\Big(r(t) + \lambda_L - \sigma_L\theta_L(t)\Big) + 0.2\pi_S(t)\Big(r(t) + \lambda_S - \sigma_S\theta_S(t)\Big) - \big(0.5\sigma_L\pi_L(t)\big)^2 - \big(0.2\sigma_S\pi_S(t)\big)^2$$

$$c_1 = \rho\frac{X(t)}{X_r(t)}$$

$$a_2 = 0.5\sigma_L\pi_L(t)$$

$$b_2 = 0.2\sigma_S\pi_S(t)$$

$$c_2 = \pi^{tp}(t)\sigma_E$$

and $C_r(t)$ denotes the bank capital adequacy ratio. Compared with the dynamic of the capital adequacy ration in Chakroun and Abid (2016), the ambiguity parameter has only impact on the drift term.

Therefore, this shows the behaviour of the dynamic of the Capital Adequacy Ration in figure 4.2.
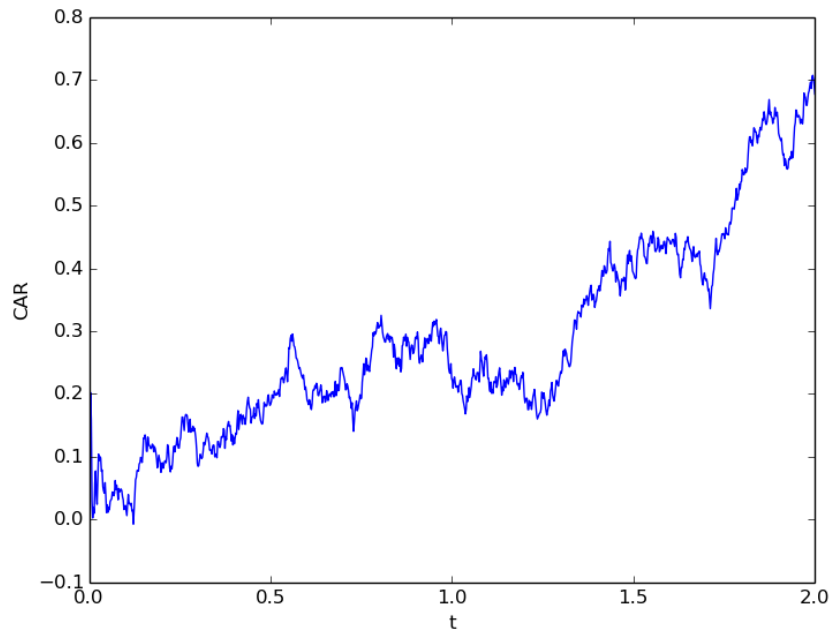
Figure 4.2: Dynamic of the Capital Adequacy Ratio(CAR)

With the initial value fixed at 21.08%, the capital adequacy ratio moves up and down over time and has a value of just over 67.75%.

4.2   Optimal portfolio choice with Choquet expectation

If the portfolio manager is ambiguity averse, the capacity c is given as $0 < c < \frac{1}{2}$ (see Roubaud et al. (2010)). Figure 4.3 is the illustration of the behaviour of the optimal proportions for a AAPM with conditional capacities.
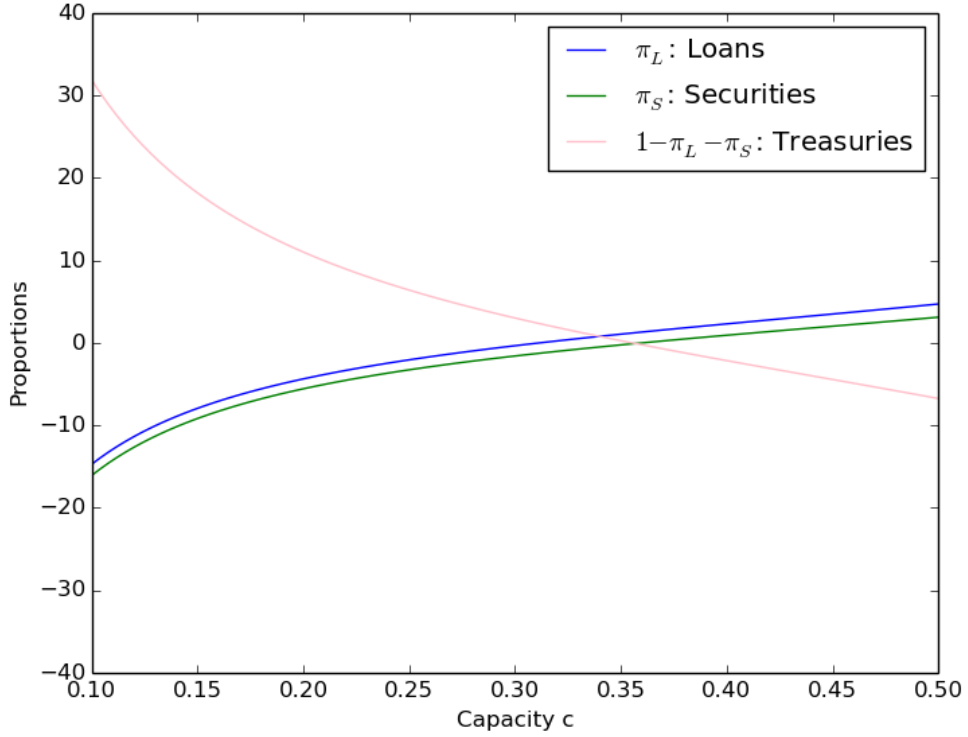
Figure 4.3: Optimal Investment Portfolio Weights modelled by Choquet Brownian motion

Figure 4.3 shows that the optimal proportions of loans and securities increase with the capacity while optimal proportions for treasuries decrease when the capacity increases.

4.2.1 Capital adequacy ratio under choquet random walk.

By the Basel III accord the risk weighted asset is given as

$$\frac{dX_r(t)}{X_r(t)} = \left[0.5\pi_L(t)\Big(r(t) + \lambda_L + \sigma_L(t)\mu\Big) + 0.2\pi_S(t)\Big(r(t) + \lambda_S + \sigma_S(t)\mu\Big)\right]dt$$

$$+0.5\sigma_L\pi_L(t)sdB_L + 0.2\sigma_S\pi_S(t)sdB_S \qquad (4.3)$$

By (3.13) and Itô formula, the derivation of the capital adequacy ratio will be given as

$$\frac{dC(t)}{X_r(t)} = \frac{C(t)}{X_r(t)}\Big[(r(t) + \pi^{tp}(t)\lambda_E)dt + \pi^{tp}(t)\sigma_E dW_E(t)\Big] - \rho\frac{Y(t)}{X_r(t)}dt$$

$$-\frac{C(t)}{X_r(t)}\left[0.5\pi_L(t)\Big(r(t) + \lambda_L + \sigma_L(t)\mu\Big) + 0.2\pi_S(t)\Big(r(t) + \lambda_S + \sigma_S(t)\mu\Big)\right]dt$$

$$+\frac{C(t)}{X_r(t)}\Big[(0.5\pi_L\sigma_L s)^2+(0.2\pi_S\sigma_S s)^2\Big]dt-\frac{C(t)}{X_r(t)}\Big[0.5\pi_L(t)\sigma_L s dB_L+0.2\pi_S(t)\sigma_S s dB_S\Big]$$

$$=\Big[C_r'(t)(a_1'-b_1')-c_1'\Big]dt-C_r'(t)\Big[a_2'dB_L+b_2'dB_S(t)-c_2'dW_E(t)\Big]$$

Where

$$a_1' = r(t)+\pi^{tp}(t)\lambda_E$$

$$b_1' = 0.5\pi_L(t)\Big(r(t)+\lambda_L+\sigma_L(t)\mu\Big)+0.2\pi_S(t)\Big(r(t)+\lambda_S+\sigma_S(t)\mu\Big)-\big(0.5\sigma_L\pi_L(t)s\big)^2-\big(0.2\sigma_S\pi_S(t)s\big)^2$$

$$c_1' = \rho\frac{Y(t)}{Y_r(t)}$$

$$a_2' = 0.5\sigma_L\pi_L(t)\mu$$

$$b_2' = 0.2\sigma_S\pi_S(t)\mu$$

$$c_2' = \pi^{tp}(t)\sigma_E$$

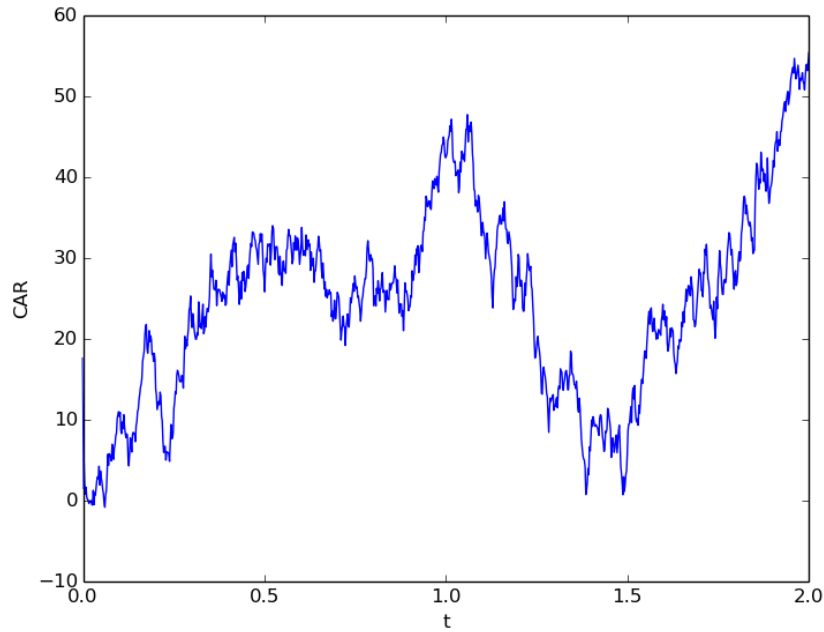and $C_r'(t)$ denotes the bank capital adequacy ratio.

Figure 4.4: Dynamic of the Capital Adequacy Ratio(CAR) for Choquet expectation

Figure 4.4 is the simulation of the capital adequacy ratio under Choquet expectation. It

shows that with the initial value fixed at 10.71%, the capital adequacy ratio moves up and

down over time and has a value of just over 59.11%.

5. Conclusion

The aim of this work was to find the optimal investment strategy for an ambiguity averse portfolio manager where the term structure of interest rate is constant by means of the classical dynamic programming and the non linear expectation. Our contribution was to obtain an optimal investment allocation strategy that optimizes the bank's asset portfolio consisting of balance sheet items of the bank.

This was achieved by constructing the total portfolio asset stochastic differential equation requiring model specification uncertainties on three categories of a bank's balance sheet items and developing the investment strategy that maximizes the bank portfolio by means of dynamic programming with a power utility function and the non linear expectation methods.

Furthermore, we derived the dynamic of the capital adequacy ratio by determining the dynamic of the risk weighted assets under Basel III agreement. We have found that for an ambiguity averse portfolio manager, the value of the capital adequacy ratio maintains the minimum required. This model is helpful in determining the optimal investment allocation strategy and the corresponding adequate capital in a financial market where the uncertainty arises.

Moreover, we model the ambiguity by means of a non linear Choquet expectation. We have been able to show that robust portfolio via worst case scenario and via Choquet expectation have completely distinct impacts on robust portfolio selection and the capital adequacy.

We conclude that while the classical approach assumes the observability of full market informations, the conditional Choquet expectation gives the more realistic assumptions in the derivation of robust portfolio selection and capital adequacy.

Since the Choquet capacity can be an unobservable process, we converted the partial information to the full information problem to get the HJB equation by the Kalman filtering technique.

Recommendations

For more research in the future, this portfolio selection problem can be further modified to take into consideration different approaches where one can use another non linear expectation and compare it with what we have done to see the best model and calibrating this model to real data would be an interesting future research endeavour.

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Appendix A. Appendix

Proof of proposition 3.3.1

Proof. Assuming that $V(t, x) = f(t)\frac{x^{1-\gamma}}{1-\gamma}$, then we have

$$V_t = f'(t)\frac{x^{1-\gamma}}{1-\gamma} \tag{A.1}$$

$$V_x = f(t)x^{-\gamma} \tag{A.2}$$

$$V_{xx} = -\gamma f(t)x^{-1-\gamma} \tag{A.3}$$

For the derivation of the worst-case drifts, we differentiate (3.21) with respect to θ_L, and θ_S, that is maximizing over $\mathcal{Q}$. Then we have

$$-\sigma_L x V_x \pi_L(t) + \frac{1}{\phi}\theta_L(t) = 0$$

$$-\sigma_S x V_x \pi_S(t) + \frac{1}{\phi}\theta_S(t) = 0.$$

This implies that

$$\theta_L^*(t) = \sigma_L \phi x V_x \pi_L(t) \tag{A.4}$$

$$\theta_S^*(t) = \sigma_S \phi x V_x \pi_S(t) \tag{A.5}$$

Replacing (A.4) and (A.5) into (3.21) gives

$$\sup_{\pi \in \mathcal{A}} \left\{ V_t + V_x x \Big[r + \pi_L(t)\Big(\lambda_L - \sigma_L^2 x V_x \phi \pi_L(t)\Big) + \pi_S(t)\Big(\lambda_S - \sigma_S^2 x V_x \phi \pi_S(t)\Big) \Big] + \frac{1}{2}V_{xx}x^2 \Big[(\pi_L(t)\sigma_L)^2 \right.$$

$$\left. + (\pi_S(t)\sigma_S)^2 \Big] + \frac{\phi}{2}\Big[\Big(\sigma_L^2 x V_x \pi_L(t)\Big)^2 + \Big(\sigma_S^2 x V_x \pi_S(t)\Big)^2 \Big] \right\} = 0 \tag{A.6}$$

For the derivation of the optimal robust investment strategy, differentiating (A.6) with respect to $\pi_L(t)$ and $\pi_S(t)$ implies that

$$xV_x\lambda_L + \sigma_L^2 x^2 \Big(V_{xx} - V_x^2\phi\Big)\pi_L(t) = 0$$

$$xV_x\lambda_S + \sigma_S^2 x^2\left(V_{xx} - V_x^2\phi\right)\pi_S(t) = 0$$

Then we have the optimal investment strategy given by

$$\pi_L^*(t) \;=\; \frac{V_x\lambda_L}{x\sigma_L^2\left(V_x^2\phi - V_{xx}\right)} \tag{A.7}$$

$$\pi_S^*(t) \;=\; \frac{V_x\lambda_S}{x\sigma_S^2\left(V_x^2\phi - V_{xx}\right)} \tag{A.8}$$

Now, substituting (A.7) and (A.8) into (A.6) yields

$$V_t + xV_x\left[r + \frac{V_x\lambda_L^2}{\sigma_L^2 x(V_x^2\phi - V_x x)}\left(1 - \frac{\phi V_x^2}{(V_x^2\phi - V_x x)}\right) + \frac{V_x\lambda_S^2}{\sigma_S^2 x(V_x^2\phi - V_x x)}\left(1 - \frac{\phi V_x^2}{(V_x^2\phi - V_x x)}\right)\right]$$
$$+\frac{1}{2}V_{xx}\left[\left(\frac{V_x\lambda_L}{\sigma_L(V_x^2\phi - V_{xx})}\right)^2 + \left(\frac{V_x\lambda_S}{\sigma_S(V_x^2\phi - V_{xx})}\right)^2\right] + \frac{\phi}{2}\left[\left(\frac{V_x^2\lambda_L}{\sigma_L(V_x^2\phi - V_x x)}\right)^2 + \left(\frac{V_x^2\lambda_S}{\sigma_S(V_x^2\phi - V_x x)}\right)^2\right] = 0$$

Which implies that

$$V_t + xV_x\left[r + \frac{V_x}{x(V_x^2\phi - V_{xx})}\left(1 - \frac{\phi V_x^2}{(V_x^2\phi - V_{xx})}\right)\left(\frac{\lambda_L^2}{\sigma_L^2} + \frac{\lambda_S^2}{\sigma_S^2}\right)\right]$$
$$+\frac{1}{2}V_{xx}\left(\frac{V_x}{(V_x^2\phi - V_{xx})}\right)^2\left(\frac{\lambda_L^2}{\sigma_L^2} + \frac{\lambda_S^2}{\sigma_S^2}\right) + \frac{\phi}{2}\left(\frac{V_x^2}{(V_x^2\phi - V_{xx})}\right)^2\left(\frac{\lambda_L^2}{\sigma_L^2} + \frac{\lambda_S^2}{\sigma_S^2}\right) = 0$$
$$\Rightarrow V_t + xV_x\left[r + A(1 - B)\left(\frac{\lambda_L^2}{\sigma_L^2} + \frac{\lambda_S^2}{\sigma_S^2}\right)\right] + \frac{1}{2}V_{xx}C^2\left(\frac{\lambda_L^2}{\sigma_L^2} + \frac{\lambda_S^2}{\sigma_S^2}\right) + \frac{\phi}{2}D^2\left(\frac{\lambda_L^2}{\sigma_L^2} + \frac{\lambda_S^2}{\sigma_S^2}\right) = 0$$
$$\Rightarrow V_t + rxV_x + \left(A(1 - B)xV_x + \frac{1}{2}C^2V_{xx} + \frac{\phi}{2}D^2\right)\left(\frac{\lambda_L^2}{\sigma_L^2} + \frac{\lambda_S^2}{\sigma_S^2}\right) = 0$$

Where

$$A \;=\; \frac{V_x}{x(V_x^2\phi - V_{xx})}$$
$$=\; \frac{f(t)x^{-\gamma}}{x\left(f^2(t)x^{-2\gamma}\phi + \gamma f(t)x^{-1-\gamma}\right)}$$
$$=\; \frac{f(t)x^{-\gamma}}{x\left(f^2(t)x^{-2\gamma}\frac{\alpha}{f}(t)x^{1-\gamma} + \gamma f(t)x^{-1-\gamma}\right)}$$
$$=\; \frac{1}{\alpha + \gamma}$$

$$B = \frac{\phi V_x^2}{(V_x^2\phi - V_{xx})}$$

$$= \frac{\alpha f^2(t)x^{-2\gamma}}{f^2(t)x^{-2\gamma}\alpha + \gamma f^2(t)x^{-2\gamma}}$$

$$= \frac{\alpha}{\alpha + \gamma}$$

Where

$$C = \frac{V_x}{(V_x^2\phi - V_{xx})}$$

$$= \frac{f(t)x^{-\gamma}}{\frac{f^2(t)x^{-2\gamma}\alpha}{\frac{(1-\gamma)f(t)x^{1-\gamma}}{1-\gamma}} + \gamma f(t)x^{-1-\gamma}}$$

$$= \frac{x}{\alpha + \gamma}$$

$$D = \frac{V_x^2}{(V_x^2\phi - V_{xx})}$$

$$= \frac{f^2(t)x^{-2\gamma}}{\frac{\alpha f^2(t)x^{-2\gamma}}{\frac{(1-\gamma)f(t)x^{1-\gamma}}{1-\gamma}} + \gamma f(t)x^{-1-\gamma}}$$

$$= \frac{f(t)x^{1-\gamma}}{\alpha + \gamma}$$

Replacing A, B, C and D by their values, we then have

$$V_t + rxV_x + \left(\frac{\lambda_L^2}{\sigma_L^2} + \frac{\lambda_S^2}{\sigma_S^2}\right)\left[\frac{1}{\alpha + \gamma}\left(1 - \frac{\alpha}{\alpha + \gamma}\right)xV_x + \frac{1}{2}\left(\frac{x}{\alpha + \gamma}\right)^2 V_{xx} + \frac{\phi}{2}\left(\frac{f(t)x^{1-\gamma}}{\alpha + \gamma}\right)^2\right] = 0$$

$$\implies V_t + rxV_x + \left(\frac{\lambda_L^2}{\sigma_L^2} + \frac{\lambda_S^2}{\sigma_S^2}\right)\left[\frac{\gamma}{\left(\alpha + \gamma\right)^2}xV_x + \frac{1}{2}\left(\frac{x}{\alpha + \gamma}\right)^2 V_{xx} + \frac{\phi}{2}\left(\frac{f(t)x^{1-\gamma}}{\alpha + \gamma}\right)^2\right] = 0$$

$$\implies V_t + rxV_x + \left(\frac{\lambda_L^2}{\sigma_L^2} + \frac{\lambda_S^2}{\sigma_S^2}\right)\frac{1}{\left(\alpha + \gamma\right)^2}\left[\gamma xV_x + \frac{1}{2}x^2 V_{xx} + \frac{\phi}{2}\left(f(t)x^{1-\gamma}\right)^2\right] = 0$$

Substituting (A.1), (A.2) and (A.3) we have the following equation

$$f'(t)\frac{x^{1-\gamma}}{1-\gamma} + rf(t)x^{1-\gamma} + \left(\frac{\lambda_L^2}{\sigma_L^2} + \frac{\lambda_S^2}{\sigma_S^2}\right)\frac{1}{\left(\alpha+\gamma\right)^2}\left[\gamma f(t)x^{1-\gamma} - \frac{\gamma}{2}f(t)x^{1-\gamma} + \frac{\alpha}{2}f(t)x^{1-\gamma}\right] = 0$$

$$\implies \frac{f'(t)}{1-\gamma} + rf(t) + \left(\frac{\lambda_L^2}{\sigma_L^2} + \frac{\lambda_S^2}{\sigma_S^2}\right)\frac{1}{\left(\alpha+\gamma\right)^2}\left(\gamma - \frac{\gamma}{2} + \frac{\alpha}{2}\right)f(t) = 0$$

$$\implies \frac{f'(t)}{1-\gamma} + \left[r + \left(\frac{\lambda_L^2}{\sigma_L^2} + \frac{\lambda_S^2}{\sigma_S^2}\right)\frac{1}{2\left(\alpha+\gamma\right)}\right]f(t) = 0$$

$$\implies f'(t) + (1-\gamma)\left[r + \frac{\left(\frac{\lambda_L^2}{\sigma_L^2} + \frac{\lambda_S^2}{\sigma_S^2}\right)}{2\left(\alpha+\gamma\right)}\right]f(t) = 0$$

Therefore we have

$$f'(t) + \beta f(t) = 0. \tag{A.9}$$

Where

$$\beta = (1-\gamma)\left[r + \frac{\left(\frac{\lambda_L^2}{\sigma_L^2} + \frac{\lambda_S^2}{\sigma_S^2}\right)}{2\left(\alpha+\gamma\right)}\right].$$

By equation (A.9) we have $f(t) = e^{\beta(T-t)}$ and $f(T) = 1$.

Therefore $V(t,x) = e^{\beta(T-t)}\frac{x^{1-\gamma}}{1-\gamma}$ and $V(T,x) = \frac{x^{1-\gamma}}{1-\gamma}$ with the optimal investment strategy given by (3.24) and (3.25). $\qquad\square$